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Application of a discrete Itô formula to determine stability (instability) of the equilibrium of a scalar linear stochastic difference equation ☆

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ABSTRACT

We apply a variant of a discretised Itô formula to develop sharp conditions for the global a.s. asymptotic stability of the equilibrium solution of a particular linear stochastic difference equation. The difference equation relies on a parameter h which can be interpreted as the stepsize of an Euler–Maruyama discretisation of a 1-dimensional linear stochastic differential equation which has constant drift and diffusion.

A natural consequence of using the discretised Itô formula is that h must be sufficiently small in order for the stability/instability conditions to be valid. However, the version of the formula developed here enables us to impose a bound on h which can be expressed explicitly in terms of the equation parameters and which is therefore computable.

1. Introduction

Consider the linear stochastic difference equation given by

$$X_{n+1} = X_n + \lambda h X_n + \mu\sqrt{h} X_n \xi_{n+1}, \quad n \in \mathbb{N}, \tag{1}$$

where $\lambda, \mu \in \mathbb{R}, h > 0$ and $\{\xi_n\}_{n\in\mathbb{N}}$ is a sequence of mutually independent $\mathcal{N}(0, 1)$ random variables defined on the complete filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_n\}_{n\in\mathbb{N}}, \mathbb{P})$. The sequence of standard Normal random variables, $\{\xi_n\}_{n\in\mathbb{N}}$, is adapted to the filtration $\{\mathcal{F}_n\}_{n\in\mathbb{N}}$ and therefore $\{X_n\}_{n\in\mathbb{N}}$ is also adapted to the filtration for any X_0. The initial value X_0 is an $\mathcal{F}_0$-measurable $\mathbb{R}$-valued random variable with finite second moment. Also Eq. (1) has an equilibrium solution at $X_n = 0$. Consider the 1-dimensional linear stochastic differential equation,

$$dX(t) = \lambda X(t)dt + \mu X(t)dB(t), \quad t \geq 0, \tag{2}$$

where $\lambda, \mu \in \mathbb{R}$ and $B(t)$ is a 1-dimensional Brownian motion defined on the complete filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$. The 1-dimensional Brownian motion, $B(t)$, is adapted to the filtration $\{\mathcal{F}_t\}_{t\geq 0}$. The initial value $X(0)$ is an $\mathcal{F}(0)$-measurable $\mathbb{R}$-valued random variable with finite second moment. Unique strong global solutions exist since (2) is a linear autonomous stochastic differential equation; see Mao [1] for details. Also note that (2) has equilibrium solution $X(t) \equiv 0$ whenever $X(0) = 0$. We can view (1) as an Euler–Maruyama discretisation of (2); see Kloeden and Platen [2] for an introduction to stochastic numerical methods. Note that the Euler–Maruyama method is a special case ($\theta = 0$) of the θ-Maruyama method which is also called the stochastic θ method.

The discrete Itô formula developed by [3] allows us to say, for h sufficiently small, when the equilibrium solution of (1) is globally a.s. asymptotically stable or unstable. This was demonstrated by [4]. The purpose of this article is to develop a

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variant of that result, which allows the computation of an explicit bound on h, and to demonstrate its use. Whenever we mention a.s. asymptotic stability throughout the paper we mean stability in a global sense.

Discrete-time analogues of the Itô formula have been developed by many other authors, for example Fujita [5] and Shiryaev [6]. These discrete analogues try to mimic the Itô formula by writing a transformation of a discrete sequence as the sum of a constant, a deterministic integral and a stochastic integral. The constant, deterministic integral and stochastic integral themselves contain a transformation of the discrete sequence. Fujita [5] showed that for a $\mathbb{Z}$-valued symmetric random walk $\{Z_n\}_{n=0,1,2,...}$ and any function $f : \mathbb{Z} \mapsto \mathbb{R}$, the transformation $f(Z_{n+1})$ can be expressed as a combination of various sums and differences of $f(Z_n)$ and $f(Z_n+K)$ where $K = -1, 1$. Shiryaev [6] showed that for any sequence of random variables $\{X_n\}_{0\le n\le N}$ defined on a probability space and an absolutely continuous function F, the transformation $F(X_n)$ can be expressed as the sum of a constant, a discrete integral, a quadratic covariation term and some remainder terms. We are particularly interested in the version provided by Appleby et al. [3] and will make certain modifications to their formula. Appleby et al. [3] considered a one-step stochastic difference equation and showed that the conditional expectation of a function, φ, of the coefficient expression with respect to a filtration can be expressed as the sum of a constant, the first and second derivatives of φ and error terms that tend to zero as h tends to zero.

Higham [7], Berkolaiko and Rodkina [8] and Higham et al. [9] used Kolmogorov's strong law of large numbers to investigate the a.s. asymptotic stability of the equilibria of stochastic difference equations that are similar to (1). In [7], the author investigates mean-square and a.s. asymptotic stability of the equilibrium of a difference equation formed when the θ-Maruyama method (referred to as the stochastic theta method) is applied to an equation similar to (2) but with complex drift and diffusion parameters. Higham states that it is difficult to determine neat characterisations of the stability regions and so relies on a mixture of analysis and numerics. Berkolaiko and Rodkina [8] considered a homogeneous equation similar to (1) with $\lambda = 0$ and $\mu = h = 1$. They were able to obtain necessary and sufficient conditions for a.s. asymptotic stability and instability of the equilibrium solution. Higham et al. [9] showed that for h sufficiently small, a.s. exponential stability of the equilibrium of (2) implies a.s. exponential stability of the equilibrium of (1). The authors generalise the results to systems of nonlinear stochastic differential equations.

Appleby et al. [3] and Berkolaiko et al. [4] used a combination of Kolmogorov's strong law of large numbers and a discrete Itô formula to obtain a.s. asymptotic stability and instability conditions. Appleby et al. [3] considered a stochastic difference equation that is similar to (1) but which has nonlinear bounded functions f and g as the drift and diffusion coefficients respectively. Berkolaiko et al. [4] considered an a.s. stability analysis of the θ-Maruyama method applied to a 2-dimensional system of stochastic differential equations which is a general case of (1). They investigated the stabilising and destabilising role of two independent noise terms and were able to derive sharp conditions of stability and instability when h is sufficiently small. In this paper we restrict our attention to (1) in order to demonstrate how the modified discrete Itô formula may be used to identify an analytic bound on h which ensures the preservation of a.s. stability/instability of the equilibrium.

In Section 2 we introduce an assumption and some necessary results that will be helpful in establishing our main results. In Section 3 we transform (1) and use Kolmogorov's strong law of large numbers to give the stability and instability conditions in terms of an expectation. In Section 4 we state and prove our version of the discrete Itô formula which will include a bound on h and explicit expressions for the error terms. In Section 5 we apply the modified discrete Itô formula to (1) and determine h sufficiently small which guarantees a.s. asymptotic stability and instability. The bound on $h > 0$ is explicit and allows us to establish necessary and sufficient conditions in terms of λ and μ which correspond to the equilibrium solution of (1) being a.s. asymptotically stable. In Section 6 we do numerical simulations that allows us to make comparisons between the expected value of the logarithm of the coefficient expression of (1) (after it is written as a one-step equation) and $h > 0$ for certain values of the equation parameters. We then compare the simulation results to those predicted by the modified discrete Itô formula.

2. Preliminaries

For the sake of completeness in the lemma below we formulate the properties of a sequence of independent standard Normal random variables.

Lemma 2.1. *Let $\{\xi_n\}_{n\in\mathbb{N}}$ be a sequence of independent standard Normal random variables, then*

(i) $\mathbb{E}(\xi_n^{2k}) = (2k-1)!!$, $\mathbb{E}(\xi_n^{2k-1}) = 0$ *for all* $k \in \mathbb{N}$;

(ii) $\mathbb{E}|\xi_n|^3 = \frac{4}{\sqrt{2\pi}} < 2$;

(iii) *If* $p(y) = \frac{1}{\sqrt{2\pi}} e^{-y^2/2}$, *then*

  (a) $p(y)|y|^4 \to 0$ *as* $|y| \to \infty$;

  (b) $p(y)|y|^4 < 1$ *when* $|y| > \left(\frac{48}{\sqrt{2\pi}}\right)^{1/2}$.

Proof. Items (i) and (ii) are standard results that follow from ξ_n being standard Normal random variables. The function $p(y)$ is the density function of each ξ_n and so it decays exponentially which accounts for item (iii)(a). Now we show item (iii)(b).