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Necessary and sufficient conditions for periodic decaying resolvents in linear discrete convolution Volterra equations and applications to ARCH(∞) processes☆

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ABSTRACT

We define a class of functions which have a known decay rate coupled with a periodic fluctuation. We identify conditions on the kernel of a linear summation convolution Volterra equation which give the equivalence of the kernel lying in this class of functions and the solution lying in this class of functions. Some specific examples are examined. In particular, this theory is used to provide a counterexample to a result regarding the rate of decay of the auto-covariance function of an ARCH(∞) process.
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1. Introduction

This paper characterises the exact decay rate of the solution of the discrete linear Volterra equation

$$X(n+1) = f(n+1) + \sum_{j=0}^{n} U(n-j)X(j), \quad n \in \mathbb{Z}^+, \qquad X(0) = X_0, \tag{1}$$

where $f : \mathbb{Z}^+ \to \mathbb{R}^d$, $U : \mathbb{Z}^+ \to \mathbb{R}^{d\times d}$ and $X_0 \in \mathbb{R}^d$. The exact rate of decay of the forcing function, f, is known, and the kernel U has known decay and periodic asymptotic behaviour. We define the associated resolvent equation of (1):

$$Z(n+1) = \sum_{j=0}^{n} U(n-j)Z(j), \quad n \in \mathbb{Z}^+, \qquad Z(0) = I, \tag{2}$$

where $Z : \mathbb{Z}^+ \to \mathbb{R}^{d\times d}$ and I is the identity matrix. By first examining (2), we can more easily analyse (1) via a variation of constants representation:

$$X(n) = Z(n)X(0) + \sum_{j=1}^{n} Z(n-j)f(j), \quad n \in \{1, 2, \ldots\}. \tag{3}$$

It is shown in [2] that, when the kernel of (1) has a particular rate of slower than exponential decay (e.g., polynomial or regularly varying decay), then the solution of (2) also has this exact rate of decay. It is from this class of weight function that the rate of decay of U in the present work is imposed. It is shown in [9,10,7] that periodicity in the kernel of perturbed summation Volterra equations implies periodicity in the solution of these equations. The stability of solutions

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of perturbed summation Volterra equations is also shown. Linear Volterra convolution and non-convolution equations are studied in [3], where conditions on the summability of the resolvent and stability of the solution are used to establish the existence of a unique bounded (in particular periodic and almost periodic) solution. Conditions guaranteeing the existence of asymptotically periodic solutions of linear non-convolution summation Volterra equations are derived in [7] via an application of admissibility theory.

Section 2 gives some fundamental definitions as well as various lemmata needed in the proof in Section 3. In Section 3, the main result establishes that the solution of (2) also decays at the same rate as the kernel and that the periodic component is preserved. This result is achieved by eliminating the effect of the periodicity, by evaluating (2) at N discrete time points, where N is the value of the period, and lifting the equation to a higher space dimension in which it is asymptotically autonomous. Then, by a careful separation of the summation term, we can form a system of equations to which we apply the admissibility theory of [2]. Moreover, it can be shown in the case when the kernel is "small" in some $\ell^1(\mathbb{Z}^+)$ sense that Z has periodic decaying asymptotic behaviour if and only if U does, and indeed both sequences can be majorised by the same weight function and possess the same period. In forthcoming work, it is planned to investigate more general forms of decay in both continuous and discrete equations, where the decay can be separated into a rate and a bounded component with some structure (such as the periodicity studied here). Lastly, in Section 4, the results developed in Section 3 are applied to demonstrate that, if a periodic fluctuation is present in the kernel of an ARCH(∞) process, then this periodic component propagates through to the auto-covariance function of the ARCH(∞) process. This example sheds further light on extant research on the memory properties of ARCH(∞) processes (see e.g., [5,8,11]).

2. Preliminary results

If d is a positive integer, the space of all $d \times d$ real matrices is denoted by $\mathbb{R}^{d\times d}$, the zero matrix by 0, and the identity matrix by I. Similarly, the space of all $d \times d$ matrices with complex-valued entries is denoted by $\mathbb{C}^{d\times d}$. A matrix $A = (A_{ij})$ in $\mathbb{R}^{d\times d}$ is *non-negative* if $A_{ij} \geq 0$, in which case we write $A \geq 0$. A partial ordering is defined on $\mathbb{R}^{d\times d}$ by letting $A \leq B$ if and only if $B - A \geq 0$. Of course $A \leq B$ and $C \geq 0$ imply that $CA \leq CB$ and $AC \leq BC$. The *absolute value* of $A = (A_{ij})$ in $\mathbb{R}^{d\times d}$ is the matrix given by $(|A|)_{ij} = |A_{ij}|$. $\mathbb{R}^{d\times d}$ can be endowed with many norms, but they are all equivalent. The *spectral radius* of a matrix A is given by $\rho(A) = \lim_{n\to\infty} \|A^n\|^{1/n}$, where $\|\cdot\|$ is any norm on $\mathbb{R}^{d\times d}$; $\rho(A)$ is independent of the norm employed to calculate it. We note that $\rho(A) \leq \rho(|A|)$. Also, if $0 \leq A \leq B$, then $\rho(A) \leq \rho(B)$. Also,

$$\rho(A) \leq \|A^k\|^{1/k}, \quad \forall k \in \mathbb{N} \tag{4}$$

In this paper, we use the matrix norm $\|A\|_\infty = \max_{1\leq i\leq N} \sum_{j=1}^{N} |A_{i,j}|$.

The set of integers is denoted by $\mathbb{Z}$, and $\mathbb{Z}^+ = \{n \in \mathbb{Z} : n \geq 0\}$. Sequences $\{u(n)\}_{n\geq 0}$ in $\mathbb{R}^d$ or $\{U(n)\}_{n\geq 0}$ in $\mathbb{R}^{d\times d}$ are sometimes identified with functions $u : \mathbb{Z}^+ \to \mathbb{R}^d$ and $U : \mathbb{Z}^+ \to \mathbb{R}^{d\times d}$. If $\{U(n)\}_{n\geq 0}$ and $\{V(n)\}_{n\geq 0}$ are sequences in $\mathbb{R}^{d\times d}$, we define the *convolution* of $\{(U * V)(n)\}_{n\geq 0}$ by $(U * V)(n) = \sum_{j=0}^{n} U(n-j)V(j)$ for $n \geq 0$. Moreover, using this definition of convolution, one may recursively define the *j-fold convolution*, $\{(U^{*j})(n)\}_{j\geq 2, n\geq 0}$, by $(U^{*2})(n) = (U * U)(n)$ and $(U^{*j})(n) = (U^{*(j-1)} * U)(n)$ for $j \geq 3$ and $n \geq 0$. The *Z-transform* of a sequence $\{U(n)\}_{n\geq 0}$ is the function in $\mathbb{C}^{d\times d}$ defined by $\tilde{U}(z) = \sum_{j=0}^{\infty} U(j)z^{-j}$, provided that z is a complex number for which the series converges absolutely. A similar definition pertains for sequences with values in other spaces.

Let $C \in \mathbb{R}^{d\times d}$. Then we say that C is a circulant matrix if $C_{i,j} = C_{d+i-j+1,1}$ for $i < j$ and $C_{i-j+1,1}$ for $i \geq j$. Such a matrix is a special type of Toeplitz matrix. We introduce a class of weight functions used throughout this paper; it is defined and studied in [2], and we state it here for completeness.

Definition 2.1. Let $r > 0$. A real-valued sequence $\gamma = \{\gamma(n)\}_{n\geq 0}$ is in $\mathcal{W}(r)$ if $\gamma(n) > 0$ for all $n \geq 0$, and

$$\lim_{n\to\infty} \frac{\gamma(n-1)}{\gamma(n)} = \frac{1}{r}, \qquad \tilde{\gamma}(r) = \sum_{i=0}^{\infty} \gamma(i) r^{-i} < \infty, \tag{5}$$

$$\lim_{m\to\infty} \left(\limsup_{n\to\infty} \frac{1}{\gamma(n)} \sum_{i=m}^{n-m} \gamma(n-i)\gamma(i) \right) = 0. \tag{6}$$

Observe that, if $r < 1$ and $\gamma \in \mathcal{W}(r)$, then γ decays, whereas, if $r > 1$, then γ diverges. Criteria for showing that a sequence $\{\gamma(n)\}_{n\geq 0}$ is in $\mathcal{W}(r)$ are given in [2]. Here, we simply note that $\gamma(n) = r^n n^{-\alpha}$ for $\alpha > 1$; $\gamma(n) = r^n n^{-\alpha} \exp(-n^\beta)$ for $\alpha \in \mathbb{R}$, $0 < \beta < 1$; and $\gamma(n) = r^n e^{-n/(\log n)}$ are all sequences in $\mathcal{W}(r)$. The sequences defined by $\gamma(n) = r^n$ and $\gamma(n) = r^n n^{-\alpha}$, $\alpha \leq 1$, are *not* in $\mathcal{W}(r)$.

In this paper, we investigate a class of kernels which have the essential rate of decay of a sequence in $\mathcal{W}(r)$, but exhibit a periodic "fluctuation" of period $N \in \mathbb{N}$ around this rate of decay. To encapsulate this idea, we give the following definition.

Definition 2.2. Let $d, N \in \mathbb{Z}^+/\{0\}$ and $r > 0$ be finite. A sequence $U = \{U(n)\}_{n\geq 0} \in \mathbb{R}^{d\times d}$ is in $\mathcal{WP}(r, N)$ if there exists a function $\phi \in \mathcal{W}(r)$ and a sequence of $d \times d$ matrices $\{A_i\}_{i=0}^{N-1}$ such that $\lim_{n\to\infty} U(Nn+i)/\phi(Nn) = A_i$. We refer to ϕ as a weight function for U.

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