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A numerical method for a class of non-linear integro-differential equations on the half line

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ABSTRACT

We design and analyse a numerical method for the solution of the following second order integro-differential boundary value problem

$$\nu(y)g(y) = \int_0^\infty k(x)g(x)dx\left(D(y)g'(y)\right)' + p(y), \qquad g'(0) = 0, \qquad \lim_{y\to+\infty} g(y) = 0,$$

which arises in the study of the kinetic theory of dusty plasmas. The method we propose represents a first insight into the numerical solution of more complicated problems and consists of a discretization of the differential and integral terms and of an iteration process to solve the resulting non-linear system. Under suitable hypotheses we prove the convergence. We will show the characteristics of the method by means of some numerical simulations.

1. Introduction

In this paper, we present a numerical method to solve the following non-linear integro-differential boundary value problem

$$\begin{cases} \nu(y)g(y) - \displaystyle\int_0^{+\infty} k(x)g(x)dx\left[D(y)g'(y)\right]' = p(y) \\ g'(0) = 0, \qquad \displaystyle\lim_{y\to+\infty} g(y) = 0, \end{cases} \qquad y \geq 0. \tag{1}$$

This problem is a representative of a class of *non-standard* integro-differential equations where the derivatives of the unknown function are multiplied by an integral term depending on the unknown itself. The problems of this class are related to some real phenomena like plasmas kinetics [1–5] and population dynamics [6] in the sense that, although (1) is still far from real life problems, it contains some important peculiarities of more complicated models. This equation is defined on the half line, and its *non-standard* nature makes the analytical and numerical study rather complicated. In [7] the same authors discuss the analytical properties of Eq. (1) and prove the existence of a solution and other additional properties which are useful in the current investigation. Nevertheless, it is worth underlining that the uniqueness remains an open problem. In this paper, we focus on the numerical solution of problem (1). In Section 2, we summarize the results contained in [7], while, in Section 3 we describe our numerical approach which consists of two steps: discretization of the differential and integral

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terms by using finite differences and a quadrature formula respectively, resolution of the non-linear system which comes out from this discretization. Section 4 is devoted to the study of the convergence of the overall method. Since problem (1) is defined on the half-line, in Section 5 we have addressed the problem to choose a sufficiently large interval $[0, T]$, with $T < +\infty$, suitable for the numerical integration. In Section 6, some numerical experiments on theoretical problems of type (1) are described and Section 7 contains some concluding remarks and some ideas about the future developments of this work.

2. Continuous problem

In this section, we briefly recall the theoretical results contained in [7] about the existence of the solution of (1) and other properties of the solution and its derivatives which will be essential for the development of the numerical analysis of the method proposed here. From now on we assume that the functions involved in (1) satisfy the following properties:

(h_1) $D \in C^1([0, +\infty))$, $\nu, k \in C^2([0, +\infty))$, $p \in C([0, +\infty))$,
(h_2) $0 < D_{\inf} \le D(y) \le D_{\sup}$, $|D'(y)| \le D_1$, $y \ge 0$,
(h_3) $0 < \nu_{\inf} \le \nu(y) \le \nu_{\sup}$, $|\nu^{(i)}(y)| \le \nu_i$, $i = 1, 2$, $y \ge 0$,
(h_4) $0 \le p(y) \le P$, $y \ge 0$,
(h_5) $\lim_{y\to+\infty} p(y) = 0$,
(h_6) $\int_0^{+\infty} p(y)dy < +\infty$,
(h_7) $k(y) \ge 0$, $y \ge 0$,
(h_8) $\int_0^{+\infty} |k^{(i)}(x)|dx < +\infty$, $i = 0, 1, 2$.

Moreover, when (h_2) holds we set

$$\overline{D} = \sup_{y\ge 0} \left| \frac{D'(y)}{D(y)} \right| < +\infty. \tag{2}$$

The theorems below report, in compact form, the results stated in Theorems 1–7 of [7] and so, the proofs are omitted.

Theorem 1. *Under the assumptions (h_1)–(h_8), there exists at least one positive solution, $g \in C^2([0, +\infty))$ of the integro-differential problem (1), bounded together with its derivatives up to order two.*

In the following, it will be useful to consider the differential problem

$$\begin{cases} \nu(y)g(y, q) - q\left[D(y)g'(y, q)\right]' = p(y) \\ g'(0, q) = 0, \quad \lim_{y\to+\infty} g(y, q) = 0, \end{cases} \quad y \ge 0, \tag{3}$$

and observe that, when

$$q = \int_0^{+\infty} k(x)g(x)dx, \tag{4}$$

it coincides with problem (1). The result below holds.

Theorem 2. *Assume that hypotheses (h_1)–(h_6) are satisfied. Then, for any fixed $q > 0$, the boundary value problem (3) has a unique non-negative solution $g(y, q) \in C^2([0, +\infty))$, which is bounded together with its derivatives up to order two.*

Now we can define, $\forall q > 0$, the function:

$$F(q) = q - \int_0^{+\infty} k(x)g(x, q)dx \tag{5}$$

where $g(x, q)$ in (5) is the solution of (3) with a fixed value of q. The next theorem holds.

Theorem 3. *Assume that hypotheses (h_1)–(h_8) are satisfied. Then, $\forall \overline{q} > 0$, $g(y, q)$ and $F(q)$ are uniformly continuous on $[\overline{q}, +\infty)$. Moreover, there exist $a, b \in (0, +\infty)$ such that $F(a)F(b) \le 0$, where*

$$a := \frac{\int_0^{+\infty} \frac{k(y)p(y)}{\nu(y)} dy}{1 + \left\| \frac{p(y)}{\nu(y)} \right\|_\infty \left(\left| \left[\frac{k}{\nu}\right]'(0) \right| D(0) + \int_0^{+\infty} \left| \left[\left(\frac{k(y)}{\nu(y)} \right)' D(y) \right]' \right| dy \right)}, \tag{6}$$

$$b := \left\| \frac{p(y)}{\nu(y)} \right\|_\infty \int_0^{+\infty} k(y)dy. \tag{7}$$