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Continuation Newton methods

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ABSTRACT

Severely nonlinear problems can only be solved by some homotopy continuation method. An example of a homotopy method is the continuous Newton method which, however, must be discretized which leads to the damped step version of Newton's method.

The standard Newton iteration method for solving systems of nonlinear equations $F(u) = 0$ must be modified in order to get global convergence, i.e. convergence from any initial point. The control of steplengths in the damped step Newton method can lead to many small steps and slow convergence. Furthermore, the applicability of the method is restricted in as much as it assumes a nonsingular and everywhere differentiable mapping $F(\cdot)$.

Classical continuation methods are surveyed. Then a new method in the form of a coupled Newton and load increment method is presented and shown to have a global convergence already from the start and second order of accuracy with respect to the load increment step and with less restrictive regularity assumptions than for the standard Newton method. The method is applied for an elastoplastic problem with hardening.

1. Introduction

Nonlinear problems arise in various contexts, such as for nonlinear boundary value problems. The variational formulation of such a problem leads to a relation like: seek $u \in U$ such that

$$a(u; u, v) = (f, v) \quad \text{holds for all } v \in V, \tag{1.1}$$

where $a(.; u, v)$ is linear in u and v. A common example is $a(u; u, v) = \int_\Omega k(u, \nabla u)\nabla u \cdot \nabla v \, d\Omega$. Here U, V are Sobolev spaces, such as $U = V = H^1(\Omega)$, where Ω is the domain of definition.

The problem is usually discretized by restricting the variation to a finite element subspace $V_h \subset V$, where we seek $u = u_h$ in a corresponding subspace U_h (where normally $U_h = V_h$) and h denotes some discretization parameter, such as a mesh width. This is a Galerkin formulation of (1.1). It leads to a nonlinear algebraic equation,

$$F(u) = b, \tag{1.2}$$

where $F(u)$, u, b are n-dimensional vectors, if V_h is spanned by n linearly independent basis functions.

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The paper is concerned with the numerical solution of nonlinear equations of the form (1.2). To compute a solution one can embed the problem in a continuous flow in the form of a differentiable solution path along some parameter (t), such as the time variable in the evolutionary problem

$$\frac{du(t)}{dt} = b - F(u(t)), \quad t > 0 \tag{1.3}$$

or, if F is differentiable,

$$\frac{d}{dt}(F(u(t))) = b - F(u(t)), \quad t > 0 \tag{1.4}$$

with a given initial value $u(0) = u_0$, and where one seeks the stationary solution, $\lim u(t),\ t \to \infty$. If F is differentiable along the solution path, then (1.4) can be rewritten in the form,

$$F'(u(t))\frac{du}{dt} = b - F(u(t)), \quad t > 0, \qquad u(0) = u_0. \tag{1.5}$$

To solve (1.5) numerically, one can use the Euler forward time-stepping method, which leads to the sequence of equations

$$F'(u^k)(u^{k+1} - u^k) = \tau(b - F(u^k)), \tag{1.6}$$

where $\tau = \tau_k = t_{k+1} - t_k > 0$ is the timestep and u^k is the corresponding approximation of $u(t_k),\ k = 0, 1, \ldots.$

If $\tau < 1$, this is the damped-step form of Newton's method to solve (1.2). As is well known, see e.g. [1–3] and Sections 2, 3 of this paper, in general a full timestep method with $\tau = 1$, does not converge unless the initial value is sufficiently close to the stationary solution. If it is sufficiently close, under certain conditions such as a Lipschitz continuous derivative $F'(\cdot)$, the method converges fast, namely superlinearly, often quadratically. To arrive at such a neighborhood of the solution one can use a damped step version of the Newton method as (1.6) at the first steps, with sufficiently small timesteps, but the control of this may lead to very small timesteps and therefore slow convergence and be costly. Furthermore, convergence is still not guaranteed because the Fréchet derivative $F'(u)$ may not exist globally or can be ill-conditioned and even indefinite causing bifurcated paths, all of which may prevent convergence or, at least, make the computations at each step very expensive.

To overcome these problems we present an alternative to the stepsize control method, which is based on an increasing load method, where the solution path is defined by

$$F(u(t)) = tb + (1-t)F(u(0)), \quad 0 < t \leq 1, \tag{1.7}$$

where $u(0)$ is the solution of the "zero" load problem.

We assume that there is a unique solution to (1.7) for each t. Then we seek the solution to the full load case,

$$F(u(1)) = b.$$

In practice, we increment the load stepwise, so we solve

$$F(\hat{u}(t_{k+1})) = t_{k+1}b + (1 - t_{k+1})F(\hat{u}(0)), \quad k = 0, 1, \ldots, \tag{1.8}$$

where $t_{k+1} = t_k + \tau_k$ and $\tau_k > 0$ are the load increment steps. This equation must be approximated which can be done by a sequence of Newton approximation steps. If just a single Newton step is used for each load case, this leads to a sequence of discrete approximations u^k, where

$$F'(u^k)(u^{k+1} - u^k) = t_{k+1}b + (1 - t_{k+1})b^0 - F(u^k), \quad k = 0, 1, \ldots$$

and where the solution $u^0 = \hat{u}(0)$ to the zero load case is assumed to be given and $b^0 = F(u^0)$.

The rational behind this method is that one gradually via better conditioned problems approaches the most difficult problem, namely the full load case. Each problem for $t < 1$ might be solvable with much less computational cost than for $t = 1$. Furthermore, as we shall see, one can readily construct a proper load step sequence without use of any adaptation. For completeness we mention here another possible approach. For nonlinear boundary value problems the space mesh parameter (h) is a natural continuation parameter. Here one solves first the nonlinear problem on a coarse mesh, interpolates the solution to the fine mesh and computes the linearized problem there just once. Under certain conditions it can be shown that the error in the corresponding approximate solution is of the same order as the discretization error on the fine mesh. Hence, the nonlinear problem is solved to sufficient accuracy essentially by just one solution of a linearized problem. However, in this paper we do not further discuss this method. For references see [4,5].

The remainder of the paper is composed as follows. In Section 2 we present a local convergence result for the classical Newton method where the linearized equations are solved to full accuracy. This enables direct error estimates in the L_2-norm of the error. However, in general it is not cost efficient to solve the linearized equations exactly.

Therefore, in Section 3 we present a method where they are solved approximately but to a controlled accuracy. In this case, we can only show convergence of the residuals, but of the errors only indirectly and depending on the norm of the inverse of the Fréchet derivative. We show also a result for the case where we have replaced the Fréchet derivative $F'(u)$, by the derivative of an approximate mapping, $K(u)$. This enables treatment of cases where F is not differentiable everywhere or ill-conditioned, which typically occurs near the solution.