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Numerical approximations in optimal control of a class of heterogeneous systems ☆

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This paper is dedicated to Svetozar Margenov on the occasion of his 60th anniversary.

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ABSTRACT

The paper presents a numerical procedure for solving a class of optimal control problems for heterogeneous systems. The latter are described by parameterized systems of ordinary differential equations, coupled by integrals along the parameter space. Such problems arise in economics, demography, epidemiology, management of biological resources, etc. The numerical procedure includes discretization and a gradient projection method for solving the resulting discrete problem. A main point of the paper is the performed error analysis, which is based on the property of metric regularity of the system of necessary optimality conditions associated with the considered problem.

1. Introduction

In principle, heterogeneous control systems, as described in [1], include age/size-structured systems, advection–reaction systems, epidemiological models for heterogeneous populations, and a variety of economic models involving agents with diverse individual features. In this paper we present a numerical approach for solving optimal control problems for such systems, focusing on the following special class of heterogeneous systems:

$$\dot{x}(t,p) = f(p, x(t,p), y(t,p), u(t,p)), \qquad x(0,p) = x_0(p), \tag{1}$$

$$y(t,p) = \int_P g(p, q, x(t,q), u(t,q))\, \mathrm{d}q. \tag{2}$$

Here $t \in [0,T]$ is interpreted as time, "dot" means differentiation with respect to t, p is a scalar parameter taking values in an interval $P = [0, \Pi]$. The state variables $x : [0,T] \times P \to \mathbf{R}^n$, $y : [0,T] \times P \to \mathbf{R}^m$, and the control variable $u : [0,T] \times P \to U \subset \mathbf{R}^r$ belong to functional spaces specified below in such a way that, together with appropriate assumptions for the functions $f : P \times \mathbf{R}^n \times \mathbf{R}^m \times \mathbf{R}^r \to \mathbf{R}^n$ and $g : P \times P \times \mathbf{R}^n \times \mathbf{R}^r \to \mathbf{R}^m$, Eqs. (1), (2) make sense for a given initial condition x_0.

We associate with system (1), (2) the following optimal control problem:

$$\min_{x,y,u} \left\{ \int_P l(x(T,p))\mathrm{d}p + \int_0^T \int_P L(p, x(t,p), y(t,p), u(t,p))\mathrm{d}p\, \mathrm{d}t \right\}, \tag{3}$$

$$u(t,p) \in U, \tag{4}$$

where l and L are scalar functions, and $U \subset \mathbf{R}^r$.

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Numerous particular optimal control models of the above type can be found in the literature (see e.g. [2,1,3] and the bibliography therein), but the main applications we have in mind are in the dynamics of populations, where p is interpreted either as a genotype projection or as some other individual-specific indicator. In these considerations $x(t, \cdot)$ represents the density of the (multi-dimensional) population along the heterogeneity space P, and y is an aggregated variable (coupling Eqs. (1) and (2) together) that represents "externalities" influencing the dynamics.

System (1), (2) does not directly cover age-structured population systems that play a crucial role in population dynamics and economics [4,5]. However, modifications of the subsequent considerations apply also to optimal control of such systems. Modifications of the presented approximation scheme and the corresponding error analysis for more general systems (such as size-structured systems or advection–reaction systems) are possible, but require additional non-straightforward work.

We mention that an explicit dependence of the data on the time t is suppressed only for simplicity. Similarly, the function g may depend on y in a sufficiently "regular" way (see [6, (A3)]). If a non-distributed control $v : [0, T] \to V$ is involved in the problem, then minor modifications are needed, as explained in Section 6.

The aim of this paper is to present a numerical procedure for solving optimal control problems of the type of (1)–(4). The numerical procedure proposed below employs the Euler discretization scheme for approximation of Pontryagin's type necessary optimality conditions for the problem. The latter involve differential equations, integral relations, and an inclusion (representing the condition of maximization of the Hamiltonian), that is, a system of generalized equations. A gradient projection technique is applied for solving this system of generalized equations. A main point of this paper is the error analysis, which provides an error estimate based on a "metric regularity assumption" for the system of optimality conditions.

We mention that second order discretization schemes, rather than Euler's one, are implemented in the software developed by the author and collaborators. The reason is explained in the discussions in Section 7. However, we base our exposition on the Euler scheme due to: (i) better readability that allows to grasp the idea; (ii) to prove a second order accuracy only on the assumption of metric regularity mentioned above is an open question.

The paper is organized as follows. In the next section we give a particular example from epidemiology. In Section 3 we present some preliminary material—assumptions, precise formulation of the problem, known optimality conditions. Section 4 presents the numerical method based on discretization and a gradient projection procedure. Section 5 is devoted to the error analysis. Some extensions and discussions are given in the two final sections.

2. An example from epidemiology

Models describing the spread of infectious diseases in heterogeneous populations are well known (see e.g. [2, Chapter 6], [3]). The one below is a typical representative, where, however, a control is involved (interpreted as prevention), thus an optimal control problem can be considered.

Below $p \in [0, \Pi] =: P$ is a scalar parameter representing a trait related to the level of risk of infection of individuals having this trait (say, intensity of risky contacts, state of the immune system, personal hygiene, or a combination of the above ones). The population has a fixed size and is divided into three groups: susceptible, infected, and recovered; $S(t, p),\ I(t, p),\ R(t, p),\ p \in P$. Here $S(t, \cdot)$ is the density of the susceptible individuals at time t, similarly for I and R. Thus $\int_P S(t, p)\,\mathrm{d}p$ is the size of the susceptible sub-population, etc. Moreover, a control $u(t, p) \in [v, 1],\ v \in (0, 1)$, is involved, interpreted as intensity of prevention applied to susceptible individuals of treat p. The dynamics of the disease is described by the following system:

$$
\begin{aligned}
&\dot S(t, p) = -\sigma(p)\, u(t, p) J(t)\, S(t, p), \qquad S(0, p) = S_0(p),\\
&\dot I(t, p) = \sigma(p)\, u(t, p) J(t)\, S(t, p) - \rho I(t, p), \qquad I(0, p) = I_0(p),\\
&\dot R(t, p) = \rho I(t, p), \qquad R(0, p) = R_0(p),\\
&J(t) = \int_P \alpha(p) I(t, p)\,\mathrm{d}p,
\end{aligned}
$$

where $\sigma(p)$ combines the strength of the disease with the specific level of risk of individuals of treat p (without prevention), ρ is the recovery rate, $\alpha(p)$ is the infectiousness of infected individuals of treat p. The prevention control reduces $\sigma(p)$ to $\sigma(p)u(t, p)$. Since the population size is obviously constant, $J(t)$ measures the infectiousness of the environment in which the susceptibles live, thus $\sigma(p)u(t, p)J(t)$ is the incidence rate if control $u(t, p)$ is applied.

Notice that the natural non-negativity of $S_0(p)$, $I_0(p)$ and $R_0(p)$, together with $u(t, p) \ge 0$, implies non-negativity of the solution $S(t, p)$, $I(t, p)$ and $R(t, p)$. The invariance of the domain $\{(S, I, R) : S \ge 0,\ I \ge 0,\ R \ge 0\}$ for any $u(t, p) \ge 0$ can be easily proved.

A reasonable objective function to be minimized is

$$
\int_0^T \int_P [\beta I(t, p) + c(p, u(t, p))\, S(t, p)]\,\mathrm{d}p\,\mathrm{d}t,
$$

where β the economic losses from one individual being infected in a unit of time, $c(p, u)$ is the per capita expenditure of applying control u to susceptibles of trait p. Typically $c(p, u),\ u \in [v, 1]$ is strongly convex and decreasing, with $c(p, 1) = 0$ (no prevention effort).