Computers and Mathematics with Applications 70 (2015) 2661–2673

Contents lists available at ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

A finite element framework for some mimetic finite difference discretizations

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ARTICLE INFO

Article history:
Available online 6 August 2015

Keywords:
Mimetic finite differences
Finite element methods
Nédélec–Raviart–Thomas finite elements
Multigrid
Local Fourier analysis

ABSTRACT

In this work we derive equivalence relations between mimetic finite difference schemes on simplicial grids and modified Nédélec–Raviart–Thomas finite element methods for model problems in $\mathbf{H}(\mathbf{curl})$ and $H(\mathrm{div})$. This provides a simple and transparent way to analyze such mimetic finite difference discretizations using the well-known results from finite element theory. The finite element framework that we develop is also crucial for the design of efficient multigrid methods for mimetic finite difference discretizations, since it allows us to use canonical inter-grid transfer operators arising from the finite element framework. We provide special Local Fourier Analysis and numerical results to demonstrate the efficiency of such multigrid methods.

1. Introduction

We consider mimetic finite difference (MFD) methods for problems in $\mathbf{H}(\mathbf{curl})$ and $H(\mathrm{div})$ with essential boundary conditions. Such methods are designed in order to have natural discrete analogues of conservation (of mass, momentum, etc.), symmetry and positivity of the operators. They are also structure preserving discretizations, namely, they form discrete de Rham complexes.

Such discretization techniques were started in the School of A. A. Samarskii at the Moscow State University, and they have been further developed and analyzed by Shashkov [1] and Vabishchevich [2]. Regarding the MFD methods, our presentation here follows Vabishchevich [2] and his Vector Analysis Grid Operators (VAGO) framework for dual simplicial/polyhedral (Delaunay/Voronoi) grids.

Many authors have contributed to the research in this field, by applying the MFD methods successfully to several applications ranging from diffusion [3–5], magnetic diffusion and electromagnetics [6] to continuum mechanics [7] and gas dynamics [8]. We refer to a recent comprehensive review paper by Lipnikov, Manzini, and Shashkov [9] and a recent book by Beirão da Veiga, Lipnikov, and Manzini [10] on the subject for details and literature review.

We are interested in the MFD discretizations of two (standard) model problems in $\mathbf{H}(\mathbf{curl})$ and $H(\mathrm{div})$. We show that the MFD methods can be fitted in a more or less standard finite element (FE) framework which leads to convergence results and makes the design of efficient and fast solvers for the resulting linear systems quite easy. Our approach is somewhat like special discrete Hodge operators and, therefore, is related to the generalized finite difference approach proposed by Bossavit

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http://dx.doi.org/10.1016/j.camwa.2015.07.012
0898-1221/

(see e.g. [11] and references therein). We point out that, in the classical finite difference setting, convergence results exist, as can be seen in [2], but deriving them is by all means not an easy task. Moreover, while we provide details on the constructions in 2D, the equivalence between the MFD methods and the FE methods carries over with trivial modifications to the 3D case as well. We have only chosen 2D because it makes the exposition much easier to understand.

Such connections between the MFD schemes and the mixed FE methods for diffusion equations with Raviart–Thomas elements have been already established, see [12–16] and references therein. In fact, designing finite element methods on arbitrary grids is a hot topic and we refer to the recent works on agglomerated grids [17–19] and virtual finite element methods [20–22].

Most of the existing works are on approximation, stability and structure preserving properties of the MFD discretizations. Developing fast solvers for the resulting linear systems is a topic that needs more attention, since the design of fast solvers makes the MFD discretizations more practical and efficient. For FE methods, solvers can be built using the agglomeration techniques introduced by Lashuk and Vassilevski [17,18]. Such techniques do not apply to the MFD discretizations (even on simplicial grids!) and, to the best of our knowledge, such results are not available in the literature. We point out though that on rectangular grids for standard finite difference schemes for $H(\text{div})$ problems, a distributive relaxation based multigrid was proposed in [23].

As we have pointed out, our goal is to apply classical multigrid and subspace correction techniques [24–28] for the mimetic discretizations, by first establishing the relation with Nédélec–Raviart–Thomas elements. Such approach automatically makes efficient methods such as the ones developed by Arnold, Falk and Winther [29] and Hiptmair and Xu (HX) [30] preconditioners applicable for the MFD methods.

Regarding the convergence of W- and V-cycle multigrid with a multiplicative Schwarz relaxation proposed in [29], we complement the numerical results with practical Local Fourier Analysis (LFA) which provides sharp estimates of the multigrid convergence rates. We use a variant of LFA that is applicable on simplicial grids (see [31]) and compare the convergence rates predicted by LFA with the actual convergence rates of W-cycle and V-cycle multigrid.

The rest of the paper is organized as follows. In Section 2, we describe the MFD schemes on simplicial grids. In Section 3 we derive the "modified" Nédélec–Raviart–Thomas FE methods and show their equivalence to the VAGO MFD schemes. Section 4 defines the multigrid components: smoothers, and, with the help of the results from Section 3, the canonical inter-grid transfer operators. In this section, we also discuss the setup and the design of appropriate LFA for edge-based discretizations and Schwarz smoothers. The results obtained from the LFA analysis are shown in Section 5, together with the convergence rates of the resulting multigrid algorithm. Finally, conclusions are drawn in Section 6.

2. Mimetic finite difference discretizations on triangular grids

We consider the following two model problems for $\mathbf{u}$ in a two dimensional simply connected domain Ω:

$$\mathbf{curl}\,\mathrm{rot}\,\mathbf{u} + \kappa\mathbf{u} = \mathbf{f}, \quad \text{in } \Omega, \tag{2.1}$$

$$-\mathbf{grad}\,\mathrm{div}\,\mathbf{u} + \kappa\mathbf{u} = \mathbf{f}, \quad \text{in } \Omega, \tag{2.2}$$

with $\kappa > 0$ and subject to essential boundary conditions (vanishing tangential or normal components respectively). We also use $\mathbf{u}$ and $\mathbf{f}$ to denote solutions and right hand sides for both problems without distinguish them in *different* equations and spaces explicitly. The corresponding variational forms (used in the derivation of the FE scheme) are: find $\mathbf{u} \in \mathbf{H}(\mathbf{curl})$ and $\mathbf{u} \in H(\text{div})$, respectively, such that

$$(\mathrm{rot}\,\mathbf{u}, \mathrm{rot}\,\mathbf{v}) + \kappa(\mathbf{u}, \mathbf{v}) = (\mathbf{f}, \mathbf{v}), \quad \text{for all } \mathbf{v} \in \mathbf{H}(\mathbf{curl}), \tag{2.3}$$

$$(\mathrm{div}\,\mathbf{u}, \mathrm{div}\,\mathbf{v}) + \kappa(\mathbf{u}, \mathbf{v}) = (\mathbf{f}, \mathbf{v}), \quad \text{for all } \mathbf{v} \in H(\text{div}). \tag{2.4}$$

In 3D we replace rot with a 3-dimensional **curl**. In the variational form, $\mathbf{H}(\mathbf{curl})$ and $H(\text{div})$, are the spaces of square integrable vector valued functions which also have square integrable rot (**curl** in 3D) or div respectively. The functions in the spaces $\mathbf{H}(\mathbf{curl})$ and $H(\text{div})$ are also assumed to satisfy the essential boundary conditions $(\mathbf{u} \times \mathbf{n}) = 0$ for (2.3) and $(\mathbf{u} \cdot \mathbf{n}) = 0$ for (2.4) where $\mathbf{n}$ is the unit normal vector outward to $\partial\Omega$.

2.1. Mimetic finite differences on a pair of dual meshes

We consider MFD schemes for (2.1) and (2.2) discretized on a pair of a primal (Delaunay) simplicial grid and a dual (Voronoi) polyhedral grid. The vertices of the Delaunay triangulation are $\{\mathbf{x}_i^D\}_{i=1}^{N_D}$, and the vertices of its dual Voronoi mesh are the circumcenters of the Delaunay triangles. We denote the Voronoi vertices by $\{\mathbf{x}_k^V\}_{k=1}^{N_V}$, and note that each such vertex corresponds to a Delaunay triangle D_k, for $k = 1, \ldots, N_V$. In Fig. 2.1 we have depicted a pair of dual meshes and marked the Delaunay grid-points by squares and the Voronoi grid-points by circles. As is typical in the MFD schemes, we assume that all triangles in the triangulation have only acute angles. This assumption guarantees that the Voronoi vertices will always be in the interior of the Delaunay triangles. For 3D analogues of this assumption we refer to [2]. By duality, to a Delaunay grid point $\mathbf{x}_i^D$, there corresponds a Voronoi polygon V_i,

$$V_i = \{\mathbf{x} \in \Omega \mid |\mathbf{x} - \mathbf{x}_i^D| \le |\mathbf{x} - \mathbf{x}_j^D|,\ j = 1, \ldots, N_D,\ j \ne i\},$$

and we denote the Voronoi edge $V_{ij} = \partial V_i \cap \partial V_j$.