Computers and Mathematics with Applications 70 (2015) 2752–2761

Contents lists available at ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

Numerical stability for nonlinear evolution equations

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ARTICLE INFO

Article history:
Available online 20 June 2015

Keywords:
Nonlinear stability
Nonlinear semigroups
Nonlinear rational approximations

ABSTRACT

The paper deals with discretisation methods for nonlinear operator equations written as abstract nonlinear evolution equations. Brezis and Pazy showed that the solution of such problems is given by nonlinear semigroups whose theory was founded by Crandall and Liggett. By using the approximation theorem of Brezis and Pazy, we show the N-stability of the abstract nonlinear discrete problem for the implicit Euler method. Motivated by the rational approximation methods for linear semigroups, we propose a more general time discretisation method and prove its N-stability as well.

1. Introduction

The abstract framework of investigating nonlinear operator equations was first introduced by Stetter in [1]. Sanz-Serna and Palencia studied general linear operator equations in [2] when it turned out that this kind of abstract framework is feasible for the stability analysis of linear evolution equations as well. They considered general one-step finite difference schemes as time discretisations, and as a special case of their results they showed the Lax–Richtmyer stability introduced in [3]. Our aim is to set this abstract framework for nonlinear problems originated from nonlinear evolution equations. We will apply nonlinear operator semigroup theory established by Crandall and Liggett in [4], Brezis and Pazy in [5] and Goldstein in [6]. Their results on nonlinear operator semigroups can be viewed as numerical approximations by implicit Euler method. In the present paper we propose a more general class of discretisations, that is, one-step methods originated from rational approximations.

Based on Fekete and Faragó [7] we introduce the abstract setting and define the natural stability concept, the N-stability for nonlinear operator equations. In Section 2 we collect the basic results in the theory of nonlinear semigroups. Section 3 is devoted to the derivation of the space and time discretisation methods needed later on, especially, the implicit Euler method and the discretisations having the same form as the rational approximation schemes for linear operators. Section 4 contains the proof of the N-stability for a special class of nonlinear operators. In Section 5 we show the correspondence with the linear stability theory presented in Sanz-Serna and Palencia [2].

1.1. Setting the problem

Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be normed spaces and $F : D(F) \subset X \to Y$ be an (possibly unbounded and nonlinear) operator. We investigate the problem

$$F(u) = 0 \quad \text{for } u \in D(F). \tag{1.1}$$

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http://dx.doi.org/10.1016/j.camwa.2015.05.023
0898-1221/

If a certain discretisation is applied to solve Eq. (1.1), one defines an index set $\mathbb{I} \subset \mathbb{N}^p$ for $p \in \mathbb{N}$, the normed spaces $(X_n, \|\cdot\|_{X_n})$, $(Y_n, \|\cdot\|_{Y_n})$, and the operator $F_n : D(F_n) \subset X_n \to Y_n$ and considers the problem

$$F_n(u_n) = 0 \quad \text{for } u_n \in D(F_n) \text{ and } n \in \mathbb{I}. \tag{1.2}$$

Throughout the paper we assume that there exist unique solutions u^* and u_n^* to both problems (1.1) and (1.2), respectively. To be able to compare the solutions u^* and u_n^*, the mappings $\varphi_n : X \to X_n$ and $\psi_n : Y \to Y_n$ need to be defined for all $n \in \mathbb{I}$. For the analysis of the discretised problem (1.2) the following notions play an important role.

Definition 1.1. (a) The discretisation scheme (1.2) is called *convergent* if

$$\lim_{n\to\infty} \|\varphi_n(u^*) - u_n^*\|_{X_n} = 0.$$

(b) The scheme is called *consistent on the element* $v \in D(F)$ if $\varphi_n(v) \in D(F_n)$, $n \in \mathbb{I}$ and

$$\lim_{n\to\infty} \|F_n(\varphi_n(v)) - \psi_n(F(v))\|_{Y_n} = 0.$$

(c) We call the scheme *N-stable* if there exists a constant $C > 0$ such that the estimate

$$\|v_n - z_n\|_{X_n} \le C\|F_n(v_n) - F_n(z_n)\|_{Y_n} \tag{1.3}$$

holds for all $v_n, z_n \in D(F_n)$ and the stability constant C is independent of n.

We remark that the limit is understood simultaneously in all indices of $\mathbb{I}$. We note that convergence follows from N-stability if the scheme is assumed to be consistent on the exact solution u^* and we further assume that

$$\lim_{n\to\infty} \|\psi_n(0) - 0\|_{Y_n} = 0.$$

In this case, namely, we have

$$\begin{aligned}\|\varphi_n(u^*) - u_n^*\|_{X_n} &\le C\|F_n(\varphi_n(u^*)) - F_n(u_n^*)\|_{Y_n}\\ &\le C\|F_n(\varphi_n(u^*)) - \psi_n(F(u^*))\|_{Y_n} + C\|\psi_n(F(u^*)) - F_n(u_n^*)\|_{Y_n},\end{aligned}$$

where the first term converges to zero as n goes to infinity due to consistency and the second term converges to zero because we have $F(u^*) = 0$ in Y and $F_n(u_n^*) = 0$ in Y_n, $n \in \mathbb{I}$. This result shows the role of both stability and consistency for obtaining convergence.

2. Nonlinear semigroups

In this section we summarise the results about the nonlinear theory we will need. Our main reference is the textbook by Ito and Kappel [8]. Let $(\mathscr{X}, \|\cdot\|_{\mathscr{X}})$ denote a Banach space. From now on we identify the operator $A : D(A) \subset \mathscr{X} \to \mathscr{X}$ with its graph in $\mathscr{X} \times \mathscr{X}$.

Definition 2.1 (*Proposition 1.8, [8]*). For $\omega \in \mathbb{R}$, an operator A on $\mathscr{X}$, i.e., $A \subset \mathscr{X} \times \mathscr{X}$, is called *$\omega$-dissipative* if for all $\tau \in \left(0, \frac{1}{|\omega|}\right)$ and $f, g \in D(A)$ we have

$$\|(I - \tau A)(f) - (I - \tau A)(g)\|_{\mathscr{X}} \ge (1 - \tau\omega)\|f - g\|_{\mathscr{X}}. \tag{2.4}$$

For $\omega = 0$ the operator A is called *dissipative*. We note that for $\omega = 0$, we have $\tau \in (0, \infty)$.

Remark 2.2 (*Proposition 1.9, [8]*). Let A be an ω-dissipative operator on $\mathscr{X}$. Then, for any $\tau \in \left(0, \frac{1}{|\omega|}\right)$, the operator $(I - \tau A)^{-1}$ is single-valued and for any $\tau \in \left(0, \frac{1}{|\omega|}\right)$ and $f, g \in \operatorname{ran}(I - \tau A)$, we have

$$\|(I - \tau A)^{-1}(f) - (I - \tau A)^{-1}(g)\|_{\mathscr{X}} \le \frac{1}{1 - \tau\omega}\|f - g\|_{\mathscr{X}}.$$

Definition 2.3 (*Definition 5.1, [8]*). Let $\mathscr{X}_0$ be a subset of $\mathscr{X}$, $\omega \in \mathbb{R}$ and $(S(t))_{t\ge 0}$ be a family of (nonlinear) operators $\mathscr{X}_0 \to \mathscr{X}_0$. The family $(S(t))_{t\ge 0}$ is called a *strongly continuous semigroup of type ω on $\mathscr{X}_0$* if it possesses the following properties.

(i) $S(0)(f) = f$ for all $f \in \mathscr{X}_0$.
(ii) $S(t + s)(f) = S(t)S(s)(f)$ for all $t, s \ge 0$ and $f \in \mathscr{X}_0$.
(iii) For any $f \in \mathscr{X}_0$ the function $(0, \infty) \ni t \to S(t)(f) \in \mathscr{X}_0$ is continuous.
(iv) There exists $\omega \in \mathbb{R}$ such that $\|S(t)(f) - S(t)(g)\|_{\mathscr{X}} \le e^{\omega t}\|f - g\|_{\mathscr{X}}$ for all $t \ge 0$ and $f, g \in \mathscr{X}_0$.