Computers and Mathematics with Applications 70 (2015) 1266–1289

Contents lists available at ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

On the algorithmic and implementational aspects of a Discontinuous Galerkin method at finite strains

Timothy J. Truster [a,b,*,1], Pinlei Chen [b,2], Arif Masud [b,3]

[a] Department of Civil and Environmental Engineering, University of Tennessee, Knoxville, 318 John D. Tickle Engineering Building, Knoxville, TN 37921, United States

[b] Department of Civil and Environmental Engineering, University of Illinois at Urbana-Champaign, 3129E Newmark Civil Engineering Laboratory, MC-250, Urbana, IL 61801-2352, United States

ARTICLE INFO

Article history:
Received 16 July 2014
Received in revised form 20 March 2015
Accepted 29 June 2015
Available online 14 August 2015

Keywords:
Discontinuous Galerkin
Stabilization
Finite strain
Variational consistency
Adjoint consistency
Variational multiscale

ABSTRACT

In this work, algorithmic modifications are proposed and analyzed for a recently developed stabilized finite strain Discontinuous Galerkin (DG) method. The distinguishing feature of the original method, referred to as VMDG, is a consistently derived expression for the numerical flux and stability tensor that account for evolving material and geometric nonlinearity in the vicinity of the interface. Herein, the proposed modifications involve simplifications to the residual force vector and tangent stiffness matrix of the VMDG method that lead to formulations similar to other existing DG methods but retain the enhanced definition for the stability parameters. The primary objective is to reduce the costs associated with implementing the method as well as executing simulations while retaining accuracy and flexibility, thereby making the formulation amenable to boarder material classes such as inelasticity. Each simplification has associated implications on the mathematical and algorithmic properties of the method, such as L^2 convergence rate, solution accuracy, continuity enforcement, and stability of the nonlinear equation solver. These implications are carefully quantified and assessed through a comprehensive numerical performance study. The range of two and three dimensional problems under consideration involves both isotropic and anisotropic materials. Both triangular and quadrilateral element types are employed along with h and p refinement. The ability of the proposed methods to produce stable and accurate results for such a broad class of problems is highlighted.

1. Introduction

Recently, a stabilized Discontinuous Galerkin (DG) method was developed by Truster et al. [1] for modeling large strain solid mechanics problems. The method, denoted herein as the Variational Multiscale Discontinuous Galerkin (VMDG) method, is consistently derived from an underlying Lagrange multiplier interface formulation and possesses a form

* Corresponding author at: Department of Civil and Environmental Engineering, University of Tennessee, Knoxville, 318 John D. Tickle Engineering Building, Knoxville, TN 37921, United States. Tel.: +1 865 974 1913; fax: +1 865 974 2669.
E-mail address: ttruster@utk.edu (T.J. Truster).
1 Assistant Professor.
2 Graduate Research Assistant.
3 Professor of Mechanics and Structures.

http://dx.doi.org/10.1016/j.camwa.2015.06.035
0898-1221/

analogous to the symmetric interior penalty Galerkin method [2]. Its novel feature is that computable expressions emerge during the course of the derivation for the stability tensor and numerical flux weighting tensors that account for geometric and material nonlinearity. While accurate and stable results were obtained for problems involving large rotations and nonconforming interfaces, the formulation is more involved to implement than the classical DG method. Namely, its consistent linearization involves the nonstandard inclusion of the sixth-order tensor of material moduli, which is the derivative of the tangent material moduli. This higher order tensor can be difficult to derive in closed-form and subsequently implement for complicated material models, particularly materials exhibiting history dependence such as plasticity. Thus, while the VMDG method is mathematically and mechanically sound, the formulation would benefit from carefully designed modifications to reduce the complexity of implementation and cost of computation while maintaining superior mathematical properties, thereby increasing its appeal to the mechanics community.

Additionally, the existing literature on theoretical and computational aspects of DG methods for nonlinear solid mechanics possesses some gaps. The fundamental theory for DG methods applied to linear PDEs has become well-established; see e.g. the analyses and references from the treatise by Arnold et al. [2]. However, theorems and analyses conducted in the linear context do not always carry over to the nonlinear context. For example, many existing DG methods for finite strains, including those for hyperelasticity [3], plasticity [4], and second-order computational homogenization [5], possess a nonsymmetric incremental weak form. Loss of symmetry has been shown in the linear context [6] to upset adjoint consistency as well as yielding suboptimal L^2 error convergence rates. While studies on adjoint consistency have been performed for nonlinear fluid mechanics [7], the suboptimal L^2 convergence has not been demonstrated or investigated for DG methods for nonlinear solid mechanics. Fewer nonlinear DG methods employ symmetric tangent matrices; examples include [1,8]. Another noteworthy symmetric DG formulation was developed by Lew and co-workers [9], for which extensive stability and accuracy analyses were conducted in [10]. In contrast to the classical interior penalty approach, their formulation employs the concept of interface lifting operators, previously analyzed for linear PDEs in [2,11]. More recently, the hybridizable DG method [12,13] has emerged which utilizes numerical traces to treat the inter-element continuity constraints. The benefit of this alternative approach is that element interior fields consisting of the displacement and strain fields can be condensed locally to leave the trace field as the only global unknown field, yielding significant cost savings. Finite strain hybrid DG formulations are presented in [14,15], where the latter was shown to possess a variational structure. However, amongst all the preceding methods, the design of the stability parameter is crucial to obtaining stable computed response, particularly in the nonlinear context. The idea of adapting [10] or evolving [1] the stability parameter with solution nonlinearity has received little attention in the literature. In summary, the preceding developments for solid mechanics DG methods would be greatly enhanced by qualitatively investigating the effects of method attributes such as symmetry and stability parameter definition upon the method performance such as accuracy and number of required Newton–Raphson iterations.

The objective of this paper is to propose a family of algorithmic modifications to the VMDG method [1] and systematically analyze their mathematical and algorithmic properties. These modifications consist of selectively neglecting terms in the interface nonlinear and incremental weak forms, resulting in both symmetric and nonsymmetric formulations reminiscent of other existing methods [3,4,8]. Throughout, emphasis is placed on analyzing how these simplifications affect the balance of ease of implementation, computational efficiency, and numerical accuracy across a range of problems. Insights into this balance are gained through carefully designed numerical studies involving large strains and rotations, isotropic and anisotropic materials, and h and p refinement. Specific measures of performance include (i) maximum convergent load level, (ii) number of Newton–Raphson iterations, (iii) cost of element integration, (iv) convergence rate in L^2 and H^1 norms with respect to analytical solutions, and (v) accuracy compared to continuous Galerkin (CG) benchmarks. Additionally, we investigate the tensorial nature of the stability tensors in the VMDG method with respect to material and geometric nonlinearity. The results of these numerical investigations provide insight into the mathematical and algorithmic properties of the VMDG method and other existing nonlinear interior penalty DG methods as well as providing a reference point for methods employing lifting operators or hybridization. Finally, the proposed simplifications enable the ready extension of the VMDG method to wider classes of materials, including inelasticity, by removing the need to evaluate the sixth-order algorithmic material moduli and by minimizing the additional computational cost at integration points.

We begin in Section 2 with a brief review of the VMDG method [1] and highlight the major equations. Then, various simplifications are proposed in Section 3 and evaluated through a comparison of numerical results in Section 4. Conclusions concerning the algorithmic attributes are drawn in Section 5.

2. Review of a finite strain stabilized discontinuous Galerkin method: VMDG

Let $\Omega \subset \mathbb{R}^{n_{sd}}$ be a mechanical body that is acted upon by a deformation $\boldsymbol{\phi}$. We denote a point in the reference configuration as $\boldsymbol{X} \in \Omega$ and its image in the current configuration as $\boldsymbol{x} \in \Omega_\phi$. The deformation gradient associated with $\boldsymbol{\phi}$ is defined by $\boldsymbol{F} = \partial \boldsymbol{x}/\partial \boldsymbol{X}$. The classical weak form of the governing equilibrium equation is stated as: Find $\boldsymbol{\phi} \in \mathcal{S}$ such that for all variations $\boldsymbol{\eta}_o \in \mathcal{V}$:

$$\int_\Omega \text{GRAD}\, \boldsymbol{\eta}_o : \boldsymbol{P}\, \text{d}V = \int_\Omega \boldsymbol{\eta}_o \cdot \rho_o \boldsymbol{f}\, \text{d}V + \int_{\Gamma_\sigma} \boldsymbol{\eta}_o \cdot \boldsymbol{T}\, \text{d}A \tag{1}$$