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Numerical solutions of weakly singular Volterra integral equations using the optimal homotopy asymptotic method

M.S. Hashmi [b,*], N. Khan [a], S. Iqbal [b]

[a] Department of Mathematics, The Islamia University of Bahawalpur, Pakistan
[b] Department of Computer Science, COMSATS Institute of Information Technology, Sahiwal Campus, Pakistan

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ABSTRACT

The purpose of this paper is to apply a numerical technique namely the optimal homotopy asymptotic method (OHAM) for finding the approximate solutions of a class of Volterra integral equations with weakly singular kernels. This method uses simple computations with quite acceptable approximate solutions, which has close agreement with exact solutions. Illustrative examples are included to demonstrate the validity and applicability of the present method and a comparison has been made with existing results.

1. Introduction

Many problems of science and engineering lead to Volterra integral equations. The singular phenomenon which appears during modeling of physical structures, is of considerable importance in mathematical physics and other branches of sciences. The weakly singular Volterra integral equations with reproducing kernels represent such phenomena that have significant applications in mathematical physics and chemical reactions including stereology, heat conduction with mixed boundary conditions [1], crystal growth, electrochemistry, superfluidity and the radiation of heat from a semi-infinite solid [2].

General form of weakly singular Volterra integral equations with reproducing kernel is [3]

$$u(x) - \int_0^x \frac{t^{\mu-1}}{x^{\mu}} k(x,t)u(t)dt = f(x), \quad x \in [0, X] \tag{1}$$

where, $f(x)$ is a given function, $u(x)$ is a function to be determined and the smooth part of the kernel $k(x,t) = 1$ can arise from diffusion problems with mixed boundary conditions. This type of equation has an infinite set of solutions, among which only one particular solution is smooth and all others are singular at $x = 0$. It has been proved that Eq. (1) has a unique solution in $C^m[0, X]$ if $\mu > 1, f \in C^m[0, X]$, the second $0 < \mu \leqslant 1,\ f \in C^1[0, X]$ (with $f(0) = 1$ for $\mu = 1$), Eq. (1) has an infinite set of solutions in $C[0, X]$, which contains only one particular solution belonging to $C^1[0, X]$ [4].

Several efficient algorithms have been proposed by researchers for $\mu > 1$. The popular methods contain the product integration methods based on Newton Cotes [1], Hermite type collocation method [5], spline collocation method and iterated collocation method [6,7], and the extrapolation algorithm [8]. But in recent years researchers have turned their attention towards solving Volterra integral equations with $0 < \mu \leqslant 1$ and have represented different methods [3,9–11]. Solutions of this class of equations have been a difficult topic to be analyzed and have received much previous investigation.

* Corresponding author.
E-mail addresses: sadiq.hashmi@gmail.com (M.S. Hashmi), nargiskhan49@gmail.com (N. Khan), shaukat.iqbal.k@gmail.com (S. Iqbal).

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Recently, numerical techniques for giving the approximate solutions based on reproductive kernel theory have been introduced [12,13].

In this paper, we articulate the concept of OHAM tenders a reasonable, reliable solutions to weakly singular integral equations based on reproductive kernels for both $\mu > 1$ and $0 < \mu \leqslant 1$. This technique was established by Marinca and Herisanu [14]. The advantage of OHAM is: built in convergence criteria which are similar to HAM but more flexible. A series of papers by Herisanu and Marinca [15] Marinca and Herisanu [16], Iqbal et al. [17] Iqbal and Javed [18] and Haq et al. [19] have proved the effectiveness, generalization and reliability of this method and obtained solutions of currently important application in science and engineering. In order to communicate the reliability of method, we deal with different examples in the subsequent section. Finally, numerical comparison between OHAM and other existing methods shows the efficiency of OHAM. Comparison graphs of exact solutions and approximate solutions are also plotted to visualize the performance of OHAM. OHAM puts forward its soundness and potential for the solutions of mentioned problems in real life applications.

2. Basic formulation of OHAM

Consider the following differential equation:

$$L\left(u(x)\right) + f(x) + N\left(u(x)\right) = 0, \qquad B\left(u, \frac{du}{dx}\right) = 0. \tag{2}$$

where L is a linear operator, $u(x)$ is an unknown function and $f(x)$ is a known function, $N(u(x))$ is a non-linear operator and B is boundary operator.

By means of OHAM one first constructs a family of equations [14]

$$(1-p)\left[L(u(x,p)) + f(x)\right] = H(p)\left[L(u(x,p)) + f(x) + N(u(x,p))\right], \qquad B\left(u(x,p), \frac{\partial u(x,p)}{\partial x}\right) = 0. \tag{3}$$

where $p \in [0, 1]$ is an embedding parameter, $H(p)$ is a non-zero auxiliary function for $p \neq 0$ and $H(0) = 0$, $u(x, p)$ is an unknown function. Obviously, when $p = 0$ and $p = 1$ it holds

$$u(x, 0) = u_0(x), \qquad u(x, 1) = u(x) \tag{4}$$

respectively. Thus, as p increases from 0 to 1, the solution $u(x, p)$ varies from $u_0(x)$ to the solution $u(x)$, where $u_0(x)$ is obtained from (2) for $p = 0$:

$$L\left(u_0(x)\right) + f(x) = 0, \qquad B\left(u_0, \frac{du_0}{dx}\right) = 0. \tag{5}$$

We choose auxiliary function $H(p)$ in the form

$$H(p) = pc_1 + p^2c_2 + p^3c_3 + \cdots. \tag{6}$$

where $c_1, c_2, \ldots$ are constants, which can be determined later. Let us consider the solution of (3) in the form

$$u(x; p, c_i) = u_0(x) + \sum_{k\geqslant 1} u_k(x, c_i)p^k, \quad i = 1, 2, \ldots \tag{7}$$

Now substituting Eq. (7) in Eq. (3) and equating the coefficients of like powers of p, we obtain the governing equations of $u_0(x)$, given by Eq. (5), and the governing equations of $u_k(x)$, i.e.

$$L\left(u_1(x)\right) = c_1 N_0\left(u_0(x)\right), \qquad B\left(u_1, \frac{du_1}{dx}\right) = 0 \tag{8}$$

$$\begin{aligned} &L\left(u_k(x) - u_{k-1}(x)\right) = c_k N_0\left(u_0(x)\right) + \sum_{i=1}^{k-1} c_i\left[L\left(u_{k-i}(x)\right) + N_{k-i}\left(u_0(x), u_1(x), \ldots, u_{k-1}(x)\right)\right] \\ &B\left(u_k, \frac{du_k}{dx}\right) = 0, \quad k = 2, 3, \ldots \end{aligned} \tag{9}$$

where $N_m\left(u_0(x), u_1(x), \ldots, u_m(x)\right)$ is the coefficient of p^m, obtained by expanding $N\left(u(x; p, c_i)\right)$ in series with respect to the embedding parameter p:

$$N\left(u(x; p, c_i)\right) = N_0\left(u_0(x)\right) + \sum_{m\geqslant 1} N_m\left(u_0, u_1, \ldots, u_m\right)p^m, \quad i = 1, 2, \ldots \tag{10}$$

where $u(x; p, c_i)$ is given by Eq. (7).

It should be emphasized that u_k for $k \geqslant 0$ are governed by the linear equations (5), (8) and (9) with the linear boundary conditions that came from the original problem, which can be easily solved. The convergence of the series Eq. (7) depends