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Some common fixed point theorems on complex valued metric spaces

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Some common fixed point theorems satisfying certain rational expressions are proved in complex valued metric spaces which generalize fixed point theorems due to Azam et al., Imdad et al. and others. Some related results are also derived besides furnishing illustrative examples to highlight the realized improvements.

1. Introduction with preliminaries

The axiomatic development of a metric space was essentially carried out by French mathematician M. Frechet in the year 1906. The utility of metric spaces in the natural growth of Functional Analysis is enormous. Inspired from the impact of this natural idea to mathematics in general and to Functional Analysis in particular, several researchers attempted various generalizations of this notion in the recent past such as: rectangular metric spaces, semimetric spaces, quasimetric spaces, quasi-semimetric spaces, pseudometric spaces, probabilistic metric spaces, 2-metric spaces, D-metric spaces, G-metric spaces, K-metric spaces, cone metric spaces etc. and by now there exists considerable literature on all these generalizations of metric spaces. For more details, one can see [1–12].

Most recently, Azam et al. [13] introduced and studied complex valued metric spaces wherein some fixed point theorems for mappings satisfying a rational inequality were established. Naturally, this new idea can be utilized to define complex valued normed spaces and complex valued inner product spaces which, in turn, offer a lot of scope for further investigation. Though complex valued metric spaces form a special class of cone metric space, yet this idea is intended to define rational expressions which are not meaningful in cone metric spaces and thus many results of analysis cannot be generalized to cone metric spaces. Indeed, the definition of a cone metric space banks on the underlying Banach space which is not a division Ring. However, in complex valued metric spaces, we can study improvements of a host of results of analysis involving divisions.

In what follows, we recall some notations and definitions that will be utilized in our subsequent discussion.

Let $\mathbb{C}$ be the set of complex numbers and $z_1, z_2 \in \mathbb{C}$. Define a partial order $\precsim$ on $\mathbb{C}$ as follows:

$$z_1 \precsim z_2 \quad \text{if and only if} \quad \mathrm{Re}(z_1) \leq \mathrm{Re}(z_2), \qquad \mathrm{Im}(z_1) \leq \mathrm{Im}(z_2).$$

Consequently, one can infer that $z_1 \precsim z_2$ if one of the following conditions is satisfied:

(i) $\mathrm{Re}(z_1) = \mathrm{Re}(z_2), \mathrm{Im}(z_1) < \mathrm{Im}(z_2)$,
(ii) $\mathrm{Re}(z_1) < \mathrm{Re}(z_2), \mathrm{Im}(z_1) = \mathrm{Im}(z_2)$,
(iii) $\mathrm{Re}(z_1) < \mathrm{Re}(z_2), \mathrm{Im}(z_1) < \mathrm{Im}(z_2)$,

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(iv) $\mathrm{Re}(z_1) = \mathrm{Re}(z_2)$, $\mathrm{Im}(z_1) = \mathrm{Im}(z_2)$.

In particular, we write $z_1 \lnsim z_2$ if $z_1 \neq z_2$ and one of (i), (ii), and (iii) is satisfied and we write $z_1 \prec z_2$ if only (iii) is satisfied. Notice that $0 \precsim z_1 \lnsim z_2 \Rightarrow |z_1| < |z_2|$, and $z_1 \preceq z_2, z_2 \prec z_3 \Rightarrow z_1 \prec z_3$.

The following definition is recently introduced by Azam et al. [13].

Definition 1. Let X be a nonempty set whereas $\mathbb{C}$ be the set of complex numbers. Suppose that the mapping $d : X \times X \to \mathbb{C}$, satisfies the following conditions:

(d_1) $0 \precsim d(x, y)$, for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$;
(d_1) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(d_1) $d(x, y) \precsim d(x, z) + d(z, y)$, for all $x, y, z \in X$.

Then d is called a complex valued metric on X, and (X, d) is called a complex valued metric space.

Definition 2. Let (X, d) be a complex valued metric space and $B \subseteq X$

(i) $b \in B$ is called an interior point of a set B whenever there is $0 \prec r \in \mathbb{C}$ such that

$$N(b, r) \subseteq B$$

where $N(b, r) = \{y \in X : d(b, y) \prec r\}$.

(ii) A point $x \in X$ is called a limit point of B whenever for every $0 \prec r \in \mathbb{C}$,

$$N(x, r) \cap (B \setminus X) \neq \emptyset.$$

(iii) A subset $A \subseteq X$ is called open whenever each element of A is an interior point of A. A subset $B \subseteq X$ is called closed whenever each limit point of B belongs to B. The family

$$F = \{N(x, r) : x \in X,\ 0 \prec r\}$$

is a sub-basis for a topology on X. We denote this complex topology by τ_c. Indeed, the topology τ_c is Hausdorff.

Definition 3. Let (X, d) be a complex valued metric space and $\{x_n\}_{n\geq 1}$ be a sequence in X and $x \in X$. We say that

(i) the sequence $\{x_n\}_{n\geq 1}$ converges to x if for every $c \in \mathbb{C}$, with $0 \prec c$ there is $n_0 \in \mathbb{N}$ such that for all $n > n_0$, $d(x_n, x) \prec c$. We denote this by $\lim_n x_n = x$, or $x_n \to x$, as $n \to \infty$,
(ii) the sequence $\{x_n\}_{n\geq 1}$ is Cauchy sequence if for every $c \in \mathbb{C}$ with $0 \prec c$ there is $n_0 \in \mathbb{N}$ such that for all $n > n_0$, $d(x_n, x_{n+m}) \prec c$,
(iii) the metric space (X, d) is a complete complex valued metric space if every Cauchy sequence is convergent.

Definition 4 (*Cf. [14]*)**.** Two families of self-mappings $\{T_i\}_{i=1}^m$ and $\{S_i\}_{i=1}^n$ are said to be pairwise commuting if:

(i) $T_iT_j = T_jT_i$, $i, j \in \{1, 2, \dots, m\}$.
(ii) $S_iS_j = S_jS_i$, $i, j \in \{1, 2, \dots, n\}$.
(iii) $T_iS_j = S_jT_i$, $i \in \{1, 2, \dots, m\}, j \in \{1, 2, \dots, n\}$.

In [13], Azam et al. established the following two lemmas.

Lemma 5 (*Cf. [13]*)**.** *Let (X, d) be a complex valued metric space and let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ converges to x if and only if $|d(x_n, x)| \to 0$ as $n \to \infty$.*

Lemma 6 (*Cf. [13]*)**.** *Let (X, d) be a complex valued metric space and let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is a Cauchy sequence if and only if $|d(x_n, x_{n+m})| \to 0$ as $n \to \infty$.*

2. Main results

Before proving our results, let us point out a fallacy (e.g. Lines 10 and 20 in page 247) in the proof of Theorem 4 of Azam et al. [13] wherein authors used: $z_1 \precsim z_2 \Rightarrow \frac{z}{z_2} \precsim \frac{z}{z_1}$ which is not a reality (e.g. $i \precsim 1 + i \Rightarrow \frac{1}{1+i} \not\precsim \frac{1}{i}$). In our results, we attempt to consolidate the proof besides improvements. Our first and following theorem generalizes Theorem 4 of Azam et al. [13].

Theorem 2.1. *If S and T are self-mappings defined on a complete complex valued metric space (X, d) satisfying the condition*

$$d(Sx, Ty) \precsim \lambda d(x, y) + \frac{\mu d(x, Sx)d(y, Ty) + \gamma d(y, Sx)d(x, Ty)}{1 + d(x, y)} \tag{2.1}$$

for all $x, y \in X$ where λ, μ, γ are nonnegative reals with $\lambda + \mu + \gamma < 1$, then S and T have a unique common fixed point.