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The evaluation of barrier option prices under stochastic volatility

Carl Chiarella[a], Boda Kang[b,*], Gunter H. Meyer[c]

[a] Finance Discipline, UTS Business School, University of Technology, Sydney, Australia

[b] Finance Discipline, UTS Business School, University of Technology, Sydney, PO Box 123, Broadway, NSW 2007, Australia

[c] School of Mathematics, Georgia Institute of Technology, Atlanta, United States

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ABSTRACT

This paper considers the problem of numerically evaluating barrier option prices when the dynamics of the underlying are driven by stochastic volatility following the square root process of Heston (1993) [7]. We develop a method of lines approach to evaluate the price as well as the delta and gamma of the option. The method is able to efficiently handle both continuously monitored and discretely monitored barrier options and can also handle barrier options with early exercise features. In the latter case, we can calculate the early exercise boundary of an American barrier option in both the continuously and discretely monitored cases.

1. Introduction

Barrier options are path-dependent options and are very popular in foreign exchange markets. They have a payoff that is dependent on the realized asset path via its level; certain aspects of the contract are triggered if the asset price becomes too high or too low during the option's life. For example, an up-and-out call option pays off the usual $\max(S-K, 0)$ at expiry unless at any time during the life of the option the underlying asset has traded at a value H or higher. In this example, if the asset reaches this level (from below, obviously) then it is said to "knock out" and become worthless. Apart from "out" options like this, there are also "in" options which only receive a payoff if a certain level is reached, otherwise they expire worthless. Barrier options are popular for a number of reasons. The purchaser can use them to hedge very specific cash flows with similar properties. Usually, the purchaser has very precise views about the direction of the market. If he or she wants the payoff from a call option but does not want to pay for all the upside potential, believing that the upward movement of the underlying will be limited prior to expiry, then he may choose to buy an up-and-out call. It will be cheaper than a similar vanilla call, since the upside is severely limited. If he is right and the barrier is not triggered he gets the payoff he wanted. The closer that the barrier is to the current asset price then the greater the likelihood of the option being knocked out, and thus the cheaper the contract.

Barrier options are common path-dependent options traded in the financial markets. The derivation of the pricing formula for barrier options was pioneered by Merton [1] in his seminal paper on option pricing. A list of pricing formulas for one-asset barrier options and multi-asset barrier options both under the geometric Brownian motion (GBM) framework can be found in the articles by Rich [2] and Wong and Kwok [3], respectively. Gao et al. [4] analyzed option contracts with both knock-out barrier and American early exercise features. Zvan et al. [5] have discussed the oscillatory behavior of the

* Corresponding author.
E-mail addresses: carl.chiarella@uts.edu.au (C. Chiarella), boda.kang@uts.edu.au (B. Kang), meyer@math.gatech.edu (G.H. Meyer).

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Crank–Nicolson method for pricing barrier options, and they applied the backward Euler method in order to avoid unwanted oscillations.

Derivative securities are commonly written on underlying assets with return dynamics that are not sufficiently well described by the GBM process proposed by Black and Scholes [6]. There have been numerous efforts to develop alternative asset return models that are capable of capturing the leptokurtic features found in financial market data, and subsequently to use these models to develop option prices that better reflect the volatility smiles and skews found in market traded options. One of the classical ways to develop option pricing models that are capable of generating such behavior is to allow the volatility to evolve stochastically, for instance according to the square-root process introduced by Heston [7]. The evaluation of barrier option prices under the Heston stochastic volatility model has been extensively discussed by Griebsch [8] in her thesis.

However, there are certain drawbacks in the evaluation of the Barrier option prices under SV using either tree or finite difference methods, these include the fact that the convergence is rather slow and it takes more effort to obtain accurate hedge ratios. Yousuf [9,10] have developed a higher order smoothing scheme for pricing barrier options under stochastic volatility. The method is stable and converges rapidly which overcome some drawbacks of the finite difference methods. But those papers do not discuss how to handle the possible early exercise features of the barrier option pricing problem.

It turns out that another well known method, the method of lines is able to overcome those disadvantages. In this paper, we introduce a unifying approach to price both continuously and discretely monitored barrier options without or with early exercise features. Specifically, except for American style knock-in options,[1] we are able to price all other kinds of European or American barrier options using the framework developed here.

The remainder of the paper is structured as follows. Section 2 outlines the problem of both continuously and discretely monitored barrier options where the underlying asset follows stochastic volatility dynamics. In Section 3 we outline the basic idea of the method of lines approach and implement it to find the price profile of the barrier option. A number of numerical examples that demonstrate the computational advantages of the method of lines approach are provided in Section 4. Finally we discuss the impact of stochastic volatility on the prices of the barrier option in Section 5 before we draw some conclusions in Section 6.

2. Problem statement-barrier option with stochastic volatility

Let $C(S, v, \tau)$ denote the price of an up-and-out (UO) call option with time to maturity τ[2] written on a stock of price S and variance v that pays a continuously compounded dividend yield q. The option has strike price K and a barrier H.

Analogously to the setting in [7], the dynamics for the share price S under the risk neutral measure are governed by the stochastic differential equation (SDE) system[3]

$$dS = (r - q)Sdt + \sqrt{v}SdZ_1, \tag{1}$$

$$dv = \kappa_v(\theta_v - v)dt + \sigma\sqrt{v}dZ_2, \tag{2}$$

where Z_1, Z_2 are standard Wiener processes and $\mathbb{E}(dZ_1 dZ_2) = \rho dt$ with $\mathbb{E}$ the expectation operator under a particular risk neutral measure. In (1), r is the risk free rate of interest. In (2) the parameter σ is the so called vol-of-vol (in fact, $\sigma^2 v$ is the variance of the variance process v). The parameters κ_v and θ_v are respectively the rate of mean reversion and long run variance of the process for the variance v. These are under the risk-neutral measure and are related to the corresponding quantities under the physical measure by a parameter that appears in the market price of volatility risk.[4]

We are also able to write down the above system (1)–(2) using independent Wiener processes. Let $W_1 = Z_2$ and $Z_1 = \rho W_1 + \sqrt{1-\rho^2}W_2$ where W_1 and W_2 are independent Wiener processes under the risk neutral measure. Then,

[1] Strictly speaking, American style knock-in options could be priced numerically as well. But the approach will be more complicated than that indicated in this paper. In fact, let us take an American up-and-in option $C_{ui}(S, v, \tau, H)$ as an example. If H is the upper barrier, then we would have

$$C_{ui}(S, v, \tau, H) = \int_0^\infty \int_0^\tau C(H, v_1, \tau - \xi)p(H, v_1, \xi | S, v)d\xi dv_1;$$

where $C(H, v_1, \tau - \xi)$ is a standard American option with stock price H, variance v_1 and time to maturity $\tau - \xi$ and $p(H, v_1, \xi | S, v)$ is the transition density (Greens function) of the two dimensional processes (S, v). Hence, we could price $C(H, v_1, \tau - \xi)$ using the method of lines for certain quadrature points on v_1 and ξ. But then we would need to work out the value of the Greens function $p(H, v_1, \xi | S, v)$ on the corresponding quadrature points as well and then evaluate the two dimensional numerical integral maybe using the sparse grid approach. Thus, it is hard to implement the detailed approach in this paper to price American-style knock in options directly.

[2] Note that $\tau = T - t$, where T is the maturity date of the option and t is the running time.

[3] Of course, since we are using a numerical technique we could in fact use more general processes for S and v. The choice of the Heston processes is driven partly by the fact that this has become a very traditional stochastic volatility model and partly because a companion paper on the evaluation of European compound options under stochastic volatility uses techniques based on a knowledge of the characteristic function for the stochastic volatility process, which is known for the Heston process (see [11]), and can be used for comparison purposes.

[4] In fact, if it is assumed that the market price of risk associated with the uncertainty driving the variance process has the form $\lambda\sqrt{v}$, where λ is a constant (this was the assumption in [7]) and $\kappa_v^{\mathbb{P}}, \theta_v^{\mathbb{P}}$ are the corresponding parameters under the physical measure, then $\kappa_v = \kappa_v^{\mathbb{P}} + \lambda\sigma$, $\theta_v = \frac{\kappa_v^{\mathbb{P}}\theta_v^{\mathbb{P}}}{\kappa_v^{\mathbb{P}} + \lambda\sigma}$. In this formulation, the choice of a risk neutral measure comes down to deciding the parameters. This could for instance be done by a calibration procedure.