Computers and Mathematics with Applications 64 (2012) 2049–2065

Contents lists available at SciVerse ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

# A numerical solution of the nonlinear controlled Duffing oscillator by radial basis functions☆

J.A. Rad [a,*], S. Kazem [b], K. Parand [a]

[a] Department of Computer Sciences, Faculty of Mathematical Sciences, Shahid Beheshti University, Evin, Tehran 19839, Iran

[b] Department of Mathematics, Imam Khomeini International University, Ghazvin 34149-16818, Iran

ARTICLE INFO

*Article history:*
Received 24 June 2011
Received in revised form 3 December 2011
Accepted 28 March 2012

*Keywords:*
Optimal control problem
Duffing oscillator
Radial basis functions
Collocation method
Gaussian RBF
Lagrange multipliers

ABSTRACT

In this research, a new numerical method is applied to investigate the nonlinear controlled Duffing oscillator. This method is based on the radial basis functions (RBFs) to approximate the solution of the optimal control problem by using the collocation method. We apply Legendre–Gauss–Lobatto points for RBFs center nodes in order to use the numerical integration method more easily; then the method of Lagrange multipliers is used to obtain the optimum of the problems. For this purpose different applications of RBFs are used. The differential and integral expressions which arise in the dynamic systems, the performance index and the boundary conditions are converted into some algebraic equations which can be solved for the unknown coefficients. Illustrative examples are included to demonstrate the validity and applicability of the technique.



## 1. Introduction

### 1.1. Introduction of the problem

Optimal control problems arise in a wide variety of disciplines. Apart from traditional areas such as aerospace engineering, robotics and chemical engineering, the optimal control theory has also been used with great success in areas as diverse as economics to biomedicine and other areas of science and has received considerable attention of researchers.

Highly accurate solutions are needed to resolve the optimal control vector in the numerical solution of optimal control problems involving nonlinear dynamical systems. However, serious analytical and numerical difficulties, such as the accumulation of roundoff truncation errors, need to be overcome before optimal control approaches find widespread practical implementation, especially for nonlinear optimal control problems.

In recent years there has been much interest in the use of spectral methods for the solution of nonlinear physical and engineering problems. The main thrust of spectral methods has been inhibited due to their lack of application to controlled nonlinear dynamical systems. The controlled nonlinear Duffing oscillator is known to describe many important oscillating phenomena in some nonlinear physical and engineering systems [1–3]

$$J = \frac{1}{2}\int_{-T}^{0} U^2(\tau)d\tau, \tag{1}$$

☆ The paper has been evaluated according to old Aims and Scope of the journal.

* Corresponding author. Tel.: +98 937 509 2738; fax: +98 21 22431650.
*E-mail addresses:* ja.amani66@gmail.com, j.amanirad@gmail.com (J.A. Rad), saeedkazem@gmail.com (S. Kazem), k_parand@sbu.ac.ir (K. Parand).

0898-1221/$ – see front matter 
doi:10.1016/j.camwa.2012.03.104

subject to

$$\ddot{X}(\tau) + \delta\dot{X}(\tau) + \omega^2 X(\tau) + \epsilon X^3(\tau) = f\cos(\alpha\tau) + U(\tau), \quad -T \le \tau \le 0, \tag{2}$$

where $T$ is known, $\omega$ is the stiffness parameter, $\delta \ge 0$ is the viscous damping coefficient, $f$ and $\alpha$ are the amplitude and frequency of the external input, respectively. Also the initial and boundary conditions are

$$X(-T) = x_0, \qquad X(0) = 0, \tag{3}$$

$$\dot{X}(-T) = x_1, \qquad \dot{X}(0) = 0. \tag{4}$$

The classical Duffing's equation was first introduced to study electronics and was published by Duffing in 1918 [4]. It is the simplest oscillator displaying catastrophic jumps of amplitude and phase when the frequency of the forcing term is taken as a gradually changing parameter. The Duffing equation has wide applications in signal processing [5], the propagation of extremely short electromagnetic pulses in a nonlinear medium [6,7], brain modeling [8], fuzzy modeling and the adaptive control of uncertain chaotic systems [9,10].

In classical development, it is well known that the variational method of optimal control theory, which typically consists of the calculus of variations and Pontryagin's methods [11], can be used to derive a set of necessary conditions that must be satisfied by an optimal control law and its associated state-control equations. These necessary optimality conditions lead to a (generally nonlinear) two-point boundary-value problem (2PBVP) that must be solved to determine an explicit expression for the optimal control. Except in some special cases, it is difficult to obtain the solution of this 2BVP and in some cases it is not practical to obtain it. In general, the solution of these distinct problems requires different numerical methods which will increase the computational time and effort.

Vlassenbroeck and Van Dooren [3] introduced a direct method for the controlled Duffing oscillator. In Vlassenbroeck et al. the state and control variables, the system dynamics, and the boundary conditions expanded in Chebyshev series of order $m$ with unknown coefficients. In order to approximate the integral in the performance index, a summation of order $m_1$ was used, and in order to compute the integrand in the performance index, a summation of order $N > m_1$ was employed. Consequently, a rather complicated system of nonlinear equations have to be solved, for the unknowns, by some kind of iterative method.

A pseudospectral collocation method for solving the nonlinear controlled Duffing oscillator is presented in [12]. This approach is based on the idea of relating Legendre–Gauss–Lobatto collocation points to the structure of orthogonal polynomials.

Elnagar and Khamayseh [13] presented an alternative computational method for solving the controlled Duffing oscillator. Their approach drew upon the power of well-developed nonlinear programming techniques and computer codes to determine the optimal solutions of nonlinear systems. Central to the idea was a proper choice of trial functions, and the distribution of the collocation points is crucial to the accuracy of the solution. Thus, they constructed the $M$th-degree interpolating polynomial using Chebyshev nodes as the collocation points and Lagrange polynomials as the trial functions to approximate the state and the control vectors.

El-Kady and Elbarbary [14] used Chebyshev polynomials for solving controlled Duffing oscillator. In [14], the control and state variables are approximated by Chebyshev series of different orders. The system dynamics, boundary conditions and performance index are approximated by using an explicit formula for the Chebyshev polynomials in terms of arbitrary order of their derivatives and a large system of nonlinear equations have to be solved.

Recently, Marzban and Razzaghi [15] have introduced an alternative computational method for solving the controlled Duffing oscillator. This method consists of reducing the controlled Duffing oscillator problem to a set of algebraic equations by expanding the second derivative of the state vector $\ddot{X}(\tau)$ and the control vector $U(\tau)$ as hybrid functions with unknown coefficients. These hybrid functions are consist of block-pulse functions and Legendre polynomials.

More recently, Lakestani et al. [16] have presented an alternative computational method for solving the controlled Duffing oscillator. Their method consists of reducing the controlled Duffing oscillator problem to a set of algebraic equations by using compactly supported linear B-spline wavelets, specially constructed for the bounded interval.

In this paper, a new computational method based on radial basis functions (RBFs) is introduced to solve the controlled Duffing oscillator. In this regard, in order to simplify the numerical integration method the Legendre–Gauss–Lobatto points for RBFs center nodes are applied, then the Lagrange multipliers method is employed to obtain optimum of the problem.

### *1.2. Introduction of the radial basis functions*

The interpolation of data on scattered points, like those obtained by mesh based schemes, such as the boundary element method (BEM) and the finite element method (FEM) is a current problem. Conventional methods such as polynomial and spline interpolations have been handled to a comprehensive range of engineering problems. Alternatively, a radial basis functions (RBFs) interpolation can be applied for such a purpose.

RBFs was first studied by Roland Hardy, an Iowa State geodesist, in 1968. This method allows scattered data to be easily used in computations. An extensive study of interpolation methods available at the time was conducted by Franke [17], and concluded that RBFs interpolations were the most accurate techniques evaluated. The theory of RBFs has originated as a means to prepare a smooth interpolation of a discrete set of data points. The concept of using RBFs for solving DEs was first