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Analysing the efficiency of some modifications of the secant method

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ABSTRACT

Some modifications of the secant method for solving nonlinear equations are revisited and the local order of convergence is found in a direct symbolic computation. To do this, a development of the inverse of the first order divided differences of a function of several variables in two points is presented. A generalisation of the efficiency index used in the scalar case to several variables is also analysed in order to use the most competitive algorithm.

1. Introduction

Let $F : D \subset \mathbb{R}^m \longrightarrow \mathbb{R}^m$ be a nonlinear function and $F \equiv (F_1, F_2, \ldots, F_m)$ with $F_i : D \subseteq \mathbb{R}^m \to \mathbb{R}$, $i = 1, 2, \ldots, m$, where D is an open convex domain in $\mathbb{R}^m$, so that we are interested in approximating a solution α of the equation

$$F(x) = 0. \tag{1}$$

An approximation of α is usually obtained by means of iterative methods. A well-known iterative method for approximating α is the secant method, which is defined by the algorithm:

$$\begin{cases} x_0, x_{-1} \quad \text{given in } D, \\ x_{n+1} = \Phi(x_{n-1}, x_n) = x_n - [x_{n-1}, x_n; F]^{-1} F(x_n), \quad n \geq 0, \end{cases} \tag{2}$$

where $[u, v; F] : D \subset \mathbb{R}^m \longrightarrow \mathbb{R}^m$ is a first order divided difference on D, given by the matrix $[u, v; F] = ([u, v; F]_{ij})_{i,j=1}^m \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^m)$, where

$$[u, v; F]_{ij} = \frac{1}{u_j - v_j} \left(F_i(u_1, \ldots, u_{j-1}, u_j, v_{j+1}, \ldots, v_m) - F_i(u_1, \ldots, u_{j-1}, v_j, v_{j+1}, \ldots, v_m) \right),$$

$i, j = 1, 2, \ldots, m$, with $u = (u_1, u_2, \ldots, u_m)$ and $v = (v_1, v_2, \ldots, v_m)$, and $\mathcal{L}(\mathbb{R}^m, \mathbb{R}^m)$ denotes the set of bounded linear functions (see [1]).

According to [1], $[u, v; F] \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^m)$ is called the divided difference of first order for the function F on u and v ($u \neq v$) if

$$[u, v; F](u - v) = F(u) - F(v), \quad u, v \in D,$$

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is satisfied. Moreover, if F is differentiable, we can define the divided difference as

$$[u, v; F](u - v) = F(u) - F(v) = \int_v^u F'(z)\, dz. \tag{3}$$

More representations of the divided difference can also be seen in [1].

An important feature of method (2) is that derivatives of the function F are not used in the algorithm, so that we can apply (2) to solve (1) if the function F is not differentiable. Another important feature of (2) is that it has superlinear convergence, since the local order of convergence is $\frac{1+\sqrt{5}}{2}$, [1].

Many variations of method (2) are analysed in the classical book by Ortega and Rheinboldt [2]. After [2], practical quasi-Newton methods, which are usually defined as

$$x_{n+1} = x_n - B_n^{-1} F(x_n), \quad n \geq 0, \tag{4}$$

for solving nonlinear systems have been extensively introduced and studied for many years. Special emphasis is given to the methods that satisfy the secant equation

$$B_{n+1} s_n = F(x_{n+1}) - F(x_n), \quad n \geq 0, \tag{5}$$

for $s_n \in \mathbb{R}^m$, at every iteration, which are usually called secant methods (see [3]). In the interesting paper [4], a family of methods is reviewed that includes Broyden's method, structured quasi-Newton methods, methods with direct updates of factorisations, row-scaling methods and column-updating methods.

According to [4], the name "quasi-Newton" was used after 1965 to describe methods of the form given by (4), where the matrices B_n are intended to be approximations of the Jacobian, such that Eq. (5) is satisfied (see [3]). Following [5], most authors call secant methods those methods that satisfy Eq. (5). Practical and theoretical perspectives of this area are also discussed in [4].

Notice that the divided difference given in (2) is fixed and only changes where it is evaluated, so that it is a prefixed matrix, while the sequence of matrices B_n in quasi-Newton methods is built in an iterated way and satisfying (5). This last situation is not considered in the methods studied in this paper, where the divided differences are prefixed in advance.

The main aim of this paper is to construct new iterative methods by modifying the method (2) with better efficiency and without using derivatives of the function F in the algorithms. To measure the efficiency of the methods, we consider a generalisation of the efficiency index in the scalar case [6], taking also into account the computational efficiency studied for systems of equations [7]. To do this, we use the order of convergence, the number of evaluations of functions and the operational cost needed in each step of the iterative methods.

We present a different way to compute the local order of convergence for iterative methods that use first order divided differences instead of derivatives. Two new algorithms are analysed and applied to solve some nonlinear systems of equations. Furthermore, we use a multi-precision arithmetic in the computations of the sequences and the approximated computational order of convergence (*ACOC*). The numerical result of *ACOC*, for each one of the iterative methods, confirms that the order of the methods is well deduced.

Finally, we present an application of the new iterative methods, where a solution of a nonlinear integral equation of mixed Hammerstein type is approximated.

2. Construction of modifications of method (2)

To obtain efficient modifications of method (2) we consider two arguments which are usually used to improve iterative methods, as we can see in [6]. When it comes to improving the efficiency of an iterative method we usually increase the order of convergence, reduce the number of evaluations of functions required in the algorithm and reduce its operational cost.

Initially, we can increase the order of convergence by composing iterative methods. In this case, if we consider $y_n = \Phi(x_{n-1}, x_n)$, we can do the following "pseudo-composition":

$$\begin{cases} x_0, x_{-1} \quad \text{given in } D, \\ y_n = \Phi(x_{n-1}, x_n), \\ x_{n+1} = \Phi(x_n, y_n), \quad n \geq 0, \end{cases} \tag{6}$$

where Φ is defined in (2). We say "pseudo-composition" as a consequence of (6) depends on two points and, consequently, we cannot compose as in one-point iterative methods [8].

Another way, which has already been used by Traub [6], consists of freezing the first order divided difference, as in the following algorithm:

$$\begin{cases} x_0, x_{-1} \quad \text{given in } D, \\ y_n = \Phi(x_{n-1}, x_n), \\ x_{n+1} = y_n - [x_{n-1}, x_n; F]^{-1} F(y_n), \quad n \geq 0, \end{cases} \tag{7}$$

where Φ is defined in (2).

In the next section, we give a development of the inverse of the first order divided difference for a function of several variables, that allows obtaining the local orders of convergence of both methods later.