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Solvability of multi-point boundary value problems for multiple term Riemann–Liouville fractional differential equations ☆

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ABSTRACT

Results on the existence of solutions of a integral type boundary value problem for the multiple term fractional differential equation with the nonlinearity depending on $D_{0+}^{\alpha}x$ are established. The analysis relies on the nonlinear alternative theory. Corollaries and examples are given to illustrate the efficiency of the main theorems.

1. Introduction

Fractional differential equations have attracted in recent years a considerable interest both in mathematics and in applications. They were used in modeling of many physical and chemical processes and in engineering. In its turn, mathematical aspects of fractional differential equations and methods of their solution were discussed by many authors.

In [1], Ahmad and Nieto considered the existence and uniqueness of solutions of the anti-periodic boundary value problem for nonlinear fractional differential equation of order $\alpha \in (1, 2]$

$$\begin{cases} \mathbf{D}_{\mathbf{0}+}^{\alpha} u(t) = f(t, u(t)), \quad t \in [0, T],\ 1 < \alpha \leq 2, \\ u(0) + u(T) = 0, \\ \mathbf{D}_{\mathbf{0}+}{}^{p} u(0) + \mathbf{D}_{\mathbf{0}+}{}^{p} u(T) = 0, \end{cases} \tag{1.1}$$

where $\mathbf{D}_{0+}^{\alpha}$ is the *Caputo's fractional derivative*, $0 < p < 1$, and $f : [0, T] \times R \to R$ is continuous. BVP (1.1) can be seen as a generalization of the anti-periodic boundary value problem.

In [2], Agarwal and Ahmad studied the solvability of the following anti-periodic boundary value problem for nonlinear fractional differential equation of order $\alpha \in (3, 4]$

$$\begin{cases} \mathbf{D}_{\mathbf{0}+}^{\alpha} u(t) = f(t, u(t)), \quad 0 < t < T,\ 3 < \alpha \leq 4, \\ u(0) + u(T) = 0, \qquad u'(0) + u'(T) = 0, \\ u''(0) + u''(T) = 0, \qquad u'''(0) + u'''(T) = 0, \end{cases} \tag{1.2}$$

where $T > 0$ is a constant, $\mathbf{D}_{0+}^{\alpha}$ is the *Caputo's fractional derivative*, f is a given continuous function defined on $[0, T] \times R$.

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Wang et al. [3] studied the existence and uniqueness of solutions of the following anti-periodic boundary value problem for nonlinear fractional differential equation of order $\alpha \in (2, 3]$

$$\begin{cases} \mathbf{D}_{\mathbf{0}+}^{\alpha} u(t) = f(t, u(t)), \quad t \in [0, 1],\ 2 < \alpha \leq 3, \\ u(0) + u(1) = 0, \qquad u'(0) + u'(1) = 0, \qquad u''(0) + u''(1) = 0, \end{cases} \tag{1.3}$$

where $\mathbf{D}_{0+}^{\alpha}$ is the *Caputo's fractional derivative*, and $f : [0, T] \times R \to R$ is continuous.

When one considers the Caputo's fractional differential equation $\mathbf{D}_{0+}^{\alpha} u(t) = 0,\ t \in (0, 1), 1 < \alpha < 2$, one gets $u(t) = a + bt$ that is continuous on $[0, 1]$, while Riemann–Liouville fractional differential equation $D_{0+}^{\alpha} u(t) = 0,\ t \in (0, 1),\ 1 < \alpha < 2$ implies $u(t) = c_1 t^{\alpha-1} + c_2 t^{\alpha-2}$. It is easy to see that u is unbounded at $t = 0$. Hence BVPs for Riemann–Liouville fractional differential equations differs from BVPs for Caputo's fractional differential equations.

However, for fractional differential equation $D_{0+}^{\alpha} u(t) = f(t),\ t \in (0, 1)$ with Riemann–Liouville fractional derivative of order $\alpha \in (1, 2)$, it is well known that

$$u(t) = -\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s)ds + c_1 t^{\alpha-1} + c_2 t^{\alpha-2}, \tag{1.4}$$

where $c_1, c_2 \in R$. One sees that u is not continuous at $t = 0$ if $c_2 \neq 0$. Hence the boundary conditions $u(0) + u(1) = 0$ and $u'(0) + u'(1) = 0$ are not applicable.

There have been many papers concerned with the existence of positive solutions of boundary value problems for fractional differential equations with Riemann–Liouville fractional derivative with multi-point or integral boundary conditions, see [4–17].

In [4], Ahmad and Nieto studied a class of anti-periodic boundary value problems for Riemann–Liouville fractional differential equations. In [18], the authors investigated the existence of solutions of the following multi-point boundary value problem for Riemann–Liouville fractional differential equation

$$\begin{cases} D_{0+}^{\alpha} u(t) = f(t, u(t), u'(t)), \quad t \in (0, 1), \\ u(1) = ku(\eta), \qquad I_{0+}^{\gamma} u(0) = 0, \end{cases} \tag{1.5}$$

where D_{0+}^{α} is Riemann–Liouville fractional derivative, f is a continuous function defined on $[0, 1] \times R^2$ and $f(t, 0, 0) \not\equiv 0$, $0 < \eta < 1$ with $k\eta^{\alpha-1} \neq 1, \gamma > 1 - \alpha$. The method used in [18] is based upon the Schauder fixed point theorem.

In [17], by using fixed point theorem in Banach space, the authors studied the existence and uniqueness of solutions of the boundary value problem for Riemann–Liouville fractional differential equation

$$\begin{cases} D_{0+}^{\alpha} u(t) = f(t, u(t), (\phi u)(t), (\psi u)(t)), \quad t \in [0, T],\ 1 < \alpha \leq 2, \\ D_{0+}^{\alpha-2} u(0^+) = 0, \qquad D_{0+} u(0^+) = \nu I_{0+}^{\alpha-1} u(\eta), \end{cases} \tag{1.6}$$

where D_{0+}^{α} is Riemann–Liouville fractional derivative, $T > 0, 0 < \eta < T, f$ is continuous on $[0, T] \times R^3$, and

$$(\phi(u)(t)) = \int_0^t \gamma(t, s)u(s)ds, \qquad (\psi(u)(t)) = \int_0^t \delta(t, s)u(s)ds$$

with γ, δ being continuous on $[0, T] \times [0, T]$.

The partial differential equations with the so-called p-Laplacian $-\mathrm{div}(|\nabla u(x)|^{p-2}\nabla u(x)) + b(x)\Phi(u(x)) = 0$ (which describes for example diffusion process, see [19]) with a spatial symmetric potential b, can be reduced to $[r(t)\Phi(y'(t))]' + c(t)\Phi(y(t)) = 0$, here $\Phi(x) = |x|^{p-2}x$ with $p > 1$. This fact leads us to study the fractional differential equations with one dimensional p-Laplacian $D_{0+}^{\beta}[\rho(t)\Phi(D_{0+}^{\alpha} u(t))] + q(t)f(t, u(t), D_{0+}^{\alpha} u(t)) = 0$. From (1.4), $\lim_{t\to 0} u(t)$ may not exist, but there exists the limit $\lim_{t\to 0} t^{2-\alpha} u(t)$.

Motivated by Ahmad and Nieto [4], Allison and Kosmatov [18], in this paper, the following multi-point boundary value problem for the nonlinear multiple term fractional differential equation with the nonlinearity depending on $D_{0+}^{\alpha} u$

$$\begin{cases} D_{0+}^{\beta}[\rho(t)\Phi(D_{0+}^{\alpha} u(t))] + q(t)f(t, u(t), D_{0+}^{\alpha} u(t)) = 0, \quad t \in (0, 1), \\ \lim\limits_{t\to 0} t^{1-\alpha} u(t) + \sum\limits_{i=1}^{m} a_i u(\xi_i) = \int_0^1 g(s, u(s), D_{0+}^{\alpha} u(s))ds, \\ \lim\limits_{t\to 0} \Phi^{-1}(t^{1-\beta}\rho(t)) D_{0+}^{\alpha} u(t) + \sum\limits_{i=1}^{m} b_i D_{0+}^{\alpha} u(\xi_i) = \int_0^1 h(s, u(s), D_{0+}^{\alpha} u(s))ds \end{cases} \tag{1.7}$$