Computers and Mathematics with Applications 64 (2012) 544–549

Contents lists available at SciVerse ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

Boundedness for commutators of n-dimensional rough Hardy operators on Morrey–Herz spaces

Guilian Gao

Department of Mathematics, Zhejiang University, 310027 Hangzhou, Zhejiang, China

ARTICLE INFO

Article history:
Received 9 September 2011
Received in revised form 13 December 2011
Accepted 16 December 2011

Keywords:
Commutator
Rough Hardy operators
Herz space
Morrey–Herz space
Besov–Lipschitz space

ABSTRACT

In this paper, we study central BMO estimates for commutators of rough Hardy operators $\mathcal{H}^b_{\Omega,\beta}$ on Morrey–Herz spaces. Furthermore, we also establish the Lipschitz estimates for $\mathcal{H}^b_{\Omega,\beta}$ on some function spaces, such as the Lebesgue spaces, the Herz spaces and the Morrey–Herz spaces.

1. Introduction

Let f be a non-negative integrable function on $\mathbb{R}^+$. The classical Hardy operator is defined by

$$Hf(x) := \frac{1}{x}\int_0^x f(t)dt, \quad x > 0.$$

Hardy's most celebrated integral inequality in [1] can be stated as follows:

$$\|Hf\|_{L^p(\mathbb{R}^+)} \le \frac{p}{p-1}\|f\|_{L^p(\mathbb{R}^+)}, \quad 1 < p < \infty. \tag{1.1}$$

Hardy's inequality has received considerable attention. Many papers involved its alternative proofs, generalizations, variants and applications. Among numerous papers about these, we choose to refer to the papers [2–5].

In 1995, Christ and Grafakos obtained the following.

Theorem 1.1 ([3]). *Let f be a locally integrable function on $\mathbb{R}^n$, $1 < p < \infty$. Then*

$$\|\mathcal{H}f\|_{L^p(\mathbb{R}^n)} \le \frac{p\nu_n}{p-1}\|f\|_{L^p(\mathbb{R}^n)},$$

where $\mathcal{H}$ is the n-dimensional Hardy operator which is defined by

$$\mathcal{H}f(x) := \frac{1}{|x|^n}\int_{|t|<|x|} f(t)dt, \quad x \in \mathbb{R}^n \setminus \{0\},$$

and the constant $p\nu_n/(p-1)$ is the best possible, $\nu_n = \pi^{n/2}/\Gamma(1+n/2)$.

E-mail address: gaoguilian305@163.com.

0898-1221/$ – see front matter © 2011 Elsevier Ltd. All rights reserved.
doi:10.1016/j.camwa.2011.12.045

In [6], Fu et al. defined the n-dimensional rough Hardy operators and its commutator as follows.

Definition 1.1. Let f be a locally integrable function on $\mathbb{R}^n$, $\beta \in \mathbb{R}^1$. The n-dimensional rough Hardy operator is defined by

$$\mathcal{H}_{\Omega,\beta}f(x) := \frac{1}{|x|^{n-\beta}} \int_{|t|<|x|} \Omega(x-t)f(t)dt, \quad x \in \mathbb{R}^n \setminus \{0\},$$

where $\Omega \in L^s(S^{n-1})$, $1 \le s < \infty$, is homogeneous of degree zero.

Definition 1.2. Let b be a locally integrable function on $\mathbb{R}^n$, $\beta \in \mathbb{R}^1$. The commutator $\mathcal{H}^b_{\Omega,\beta}$ is defined by

$$\mathcal{H}^b_{\Omega,\beta}f(x) := \frac{1}{|x|^{n-\beta}} \int_{|t|<|x|} (b(x)-b(t))\Omega(x-t)f(t)dt,$$

where $\Omega \in L^s(S^{n-1})$, $1 \le s < \infty$, is homogeneous of degree zero.

For simplicity, we denote by $\mathcal{H}_{\Omega,\beta}$ the rough fractional Hardy operator when $0 < \beta < n$, and the standard rough Hardy operator when $\beta = 0$(i.e.,$\mathcal{H}_{\Omega,0} = \mathcal{H}_{\Omega}$). Also we denote $\mathcal{H}^b_{\Omega,0} = \mathcal{H}^b_{\Omega}$. When $\Omega \equiv 1$, we write $\mathcal{H}_{\Omega,\beta} = \mathcal{H}_{\beta}$ and $\mathcal{H}^b_{\Omega,\beta} = \mathcal{H}_{\beta,b}$.

And in [6], the authors have given that the commutator $\mathcal{H}^b_{\Omega,\beta}$ is bounded from $\dot{K}^{\alpha,p_1}_{q_1}(\mathbb{R}^n)$ to $\dot{K}^{\alpha,p_2}_{q_2}(\mathbb{R}^n)$ if $b \in C\dot{M}O^{\max\{q_2,u\}}$ and $\alpha < \frac{n}{u}$, where $1 < q_1, q_2 < \infty$, $\frac{1}{q_2} = \frac{1}{q_1} - \frac{\beta}{n}$, $0 < p_1 \le p_2 < \infty$, $q_1' < s \le \infty$, and $\frac{1}{u} = \frac{1}{q_1'} - \frac{1}{s}$. In this paper, we will study the boundedness of $\mathcal{H}^b_{\Omega,\beta}$ on Morrey–Herz spaces. On the other hand, Zheng and Fu in [7] have proved the commutator $\mathcal{H}_{\beta,b}$ which is generated by the n-dimensional fractional Hardy operator $\mathcal{H}_{\beta}$ and $b \in \dot{\Lambda}_{\gamma}(\mathbb{R}^n)$ is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$, where $0 < \gamma < 1$, $1 < p, q < \infty$ and $\frac{1}{p} - \frac{1}{q} = \frac{\gamma+\beta}{n}$. Furthermore, the boundedness of $\mathcal{H}_{\beta,b}$ on the homogeneous Herz space $\dot{K}^{\alpha,p}_{q}$ is obtained. A natural question is whether commutators of n-dimensional rough Hardy operators also have boundedness on these spaces. The answer is affirmative. The main purpose of this paper is to generalize the above results to the case of Morrey–Herz space.

In general, we will not be interested in obtaining the best constants in the inequalities but in the boundedness of commutators. C will often be used to denote a constant, but C may not be the same constant from one occurrence to the next. Let q' be the conjugate index of q whenever $q \ge 1$, i.e., $\frac{1}{q} + \frac{1}{q'} = 1$.

2. Definitions

Let us first recall the definition of the homogeneous central BMO spaces.

Definition 2.1 (*[8]*). Let $1 \le q < \infty$. $C\dot{M}O^q(\mathbb{R}^n)$ is the space of all functions $f \in L^q_{\mathrm{loc}}(\mathbb{R}^n)$ satisfying

$$\|f\|_{C\dot{M}O^q(\mathbb{R}^n)} = \sup_{r>0}\left(\frac{1}{|B(0,r)|}\int_{B(0,r)} |f(x) - f_{B(0,r)}|^q dx\right)^{\frac{1}{q}} < \infty,$$

where $B(0,r) = \{x \in \mathbb{R}^n : |x| < r\}$ and $f_{B(0,r)}$ is the mean value of f on $B(0,r)$.

The spaces $C\dot{M}O^q(\mathbb{R}^n)$ are first introduced by Lu and Yang in the study of dual spaces of the homogeneous Herz type Hardy spaces. Obviously, $\mathrm{BMO}(\mathbb{R}^n) \subsetneqq C\dot{M}O^q(\mathbb{R}^n)$ for all $1 \le q < \infty$, and $C\dot{M}O^q(\mathbb{R}^n) \subsetneqq C\dot{M}O^p(\mathbb{R}^n)$, $1 \le p < q < \infty$. For $k \in \mathbb{Z}$, let $B_k = \{x \in \mathbb{R}^n : |x| \le 2^k\}$, $C_k = B_k \setminus B_{k-1}$, and $\chi_k (k \in \mathbb{Z})$ denote the characteristic function of the set C_k.

Definition 2.2 (*[9]*). Let $\alpha \in \mathbb{R}$, $0 < p \le \infty$ and $0 < q < \infty$. The homogeneous Herz space $\dot{K}^{\alpha,p}_{q}(\mathbb{R}^n)$ is defined by

$$\dot{K}^{\alpha,p}_{q}(\mathbb{R}^n) = \{f \in L^q_{\mathrm{loc}}(\mathbb{R}^n \setminus \{0\}) : \|f\|_{\dot{K}^{\alpha,p}_{q}(\mathbb{R}^n)} < \infty\}$$

where

$$\|f\|_{\dot{K}^{\alpha,p}_{q}(\mathbb{R}^n)} = \left(\sum_{k=-\infty}^{\infty} 2^{k\alpha p}\|f\chi_k\|^p_{L^q(\mathbb{R}^n)}\right)^{\frac{1}{p}}$$

with the usual modification made when $p = \infty$.

Obviously, $\dot{K}^{0,q}_{q}(\mathbb{R}^n) = L^q(\mathbb{R}^n)$ and $\dot{K}^{\alpha/q,q}_{q}(\mathbb{R}^n) = L^q(\mathbb{R}^n, |x|^{\alpha})$, the Lebesgue spaces with power weight $|x|^{\alpha}$.