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An efficient direct solver for multidimensional elliptic Robin boundary value problems using a Legendre spectral-Galerkin method

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ABSTRACT

In this paper, a Legendre–Galerkin method for solving second-order elliptic differential equations subject to the most general nonhomogeneous Robin boundary conditions is presented. The homogeneous Robin boundary conditions are satisfied exactly by expanding the unknown variable using a polynomial basis of functions which are built upon the Legendre polynomials. The direct solution algorithm here developed for the homogeneous Robin problem in two-dimensions relies upon a tensor product process. Nonhomogeneous Robin data are taken into account by means of a lifting. Such a lifting is performed in two successive steps, the first one to account for the data specified at the corners and the second one to account for the boundary values prescribed in the interior of the sides. Numerical results indicating the high accuracy and effectiveness of these algorithms are presented.

1. Introduction

Spectral methods are a widely used tool in the solution of differential equations [1], function approximation and variational problems [2–5]. They involve representing the solution to a problem in terms of truncated series of smooth global functions. They give very accurate approximations for a smooth solution with relatively few degrees of freedom. This accuracy comes about because the spectral coefficients, f_n, typically tend to zero faster than any algebraic power of their index n, showing either exponential or sometimes super-exponential convergence [6]. On the non periodic canonical interval $[-1, 1]$, the Jacobi polynomials are a well-known class of polynomials exhibiting spectral convergence [7–10], of which particular examples are Chebyshev polynomials of the first kind, Chebyshev polynomials of the second kind [11], and Legendre polynomials [12–14]. Chebyshev polynomials of the first kind are equal-ripple (uniform oscillations) and those of the second kind are equal-area (the area under the curve between any two consecutive zeros is constant). Lastly, Legendre polynomials minimize the error between any function and its approximation in the L^2-norm.

Finding a fast and accurate solution of elliptic equations is often an important step in the process of solving problems of fluid dynamics and in other scientific computing applications [15,3,4]. Elliptic equations almost always involve known boundary conditions which can be fully exploited in a Galerkin method [16,1,17,18]. Such a scheme adopts an expansion in terms of a global basis set constructed so that each member explicitly satisfies the boundary conditions. By encoding this additional information, out of all numerical methods, this approach almost always provides the most suitable numerical representation. If an analytic solution of a differential equation is known but difficult to compute, it is expedient to write it in terms of a spectral expansion (for instance in Legendre polynomials) which, once the coefficients are known, is easy to evaluate. In this paper, we shall see such an approximation method can be used to extend the results of [7,19] to elliptic equations with nonhomogeneous Robin boundary conditions in two dimensions.

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Doha and Abd-Elhameed [20] proposed and applied a spectral tau method based on expansion in doubly ultraspherical polynomials for the parabolic and elliptic partial differential equations subject to the most general nonhomogeneous mixed boundary conditions. In [7], the authors presented some efficient Jacobi–Galerkin algorithms for direct solution of second-order differential equations subject to homogeneous Dirichlet boundary conditions based on the matrix decomposition (diagonalization) method [7,21]; while in [19], fast Jacobi–Galerkin algorithms for solving one- and two-dimensional elliptic equations with Neumann boundary conditions are considered.

Auteri et al. [16] introduced an efficient Legendre–Galerkin direct spectral solver for the Neumann problem associated with Laplace and Helmholtz operators in two dimensions that uses a double diagonalization process very similar to that of the Dirichlet spectral solver [22]. The method of [16] is also similar to that of [22] in the way that nonzero boundary values are taken into account. Both methods use a lifting of the boundary data using a two-step procedure, the first step assigning suitable values at the corners and the second assigning the boundary data on the four edges. The point values in the corners will be shown to stem from the derivative of the Neumann datum and are associated with the presence of compatibility conditions between the two slopes in each corner of the domain, as explored by the analysis of Grisvard [23]. Furthermore, Bialecki and Karageorghis [24] proposed a spectral collocation method based on Legendre polynomials for solving the Helmholtz equation in two-dimensions subject to nonhomogeneous Robin boundary conditions. The algorithm in the present work is somewhat related to the ideas used by Auteri and Quartapelle [22], Auteri et al. [16], Doha and Bhrawy [7,8], Doha et al. [19] and Shen [18] in developing fast algorithms for various purposes.

In this paper, we are concerned with the direct solution techniques for second-order elliptic equations subject to nonhomogeneous Robin boundary conditions, using the Legendre–Galerkin approximations (LGM). We present appropriate Legendre basis functions for the approximation of an ordinary differential operator and give the explicit representation of the spectral matrix of the second-order derivative as well as of the mass matrix, including the modes required to impose nonhomogeneous boundary conditions. The direct solution algorithms here developed for the homogeneous Robin problem in two-dimensions rely upon a tensor product process [8,19]. Moreover the treatment of the nonhomogeneous Robin boundary data over a rectangular domain is described, by recalling the concept of lifting the nonzero boundary values. More precisely, such a lifting is performed in two successive steps, the first one to account for the data specified at the corners, in general this step is cumbersome, and the second one to account for the boundary values prescribed in the interior of the sides. The structure of this lifting is similar to that of the two-step procedure proposed in [22] for the Dirichlet boundary value problem and in [16,19] for the Neumann boundary value problem. Numerical results are presented in which the usual exponential convergence behavior of spectral approximations is exhibited.

The content of the paper is organized as follows. In Section 2 we start by introducing the basic concepts. In particular, in Section 2.1 the construction of the basis functions; in Section 2.2 the spectral mass and stiffness matrices are presented and we discuss an algorithm for solving the second-order one-dimensional elliptic equations subject to nonhomogeneous Robin boundary conditions. In Section 3 we describe how problems in two-dimensions with nonhomogeneous Robin boundary conditions can be efficiently transformed into problems with homogeneous Robin boundary conditions. In Section 4, we present various numerical results exhibiting the accuracy and efficiency of our numerical algorithms. We end the paper with a few concluding remarks in Section 5.

2. 1-*D* problem with Robin conditions

In this section, we consider the following one dimensional model problem:

$$\gamma_1 u - u_{xx} = f(x), \quad \text{in } I = (-1, 1), \tag{2.1}$$

with the Robin type boundary condition

$$\begin{aligned} &a_+ u(1) + b_+ u_x(1) = e_+, \\ &a_- u(-1) + b_- u_x(-1) = e_-, \end{aligned} \tag{2.2}$$

where the given constants $a_+,\ b_+,\ a_-,\ b_-$ are such that

$$(a_+ + 2b_+)a_- - (2a_+ + 3b_+)b_- \neq 0,$$

while $\gamma_1 > 0$ if $a_+ = a_- = 0$ and $\gamma_1 \geq 0$ otherwise, and $f(x)$ is a given source function.

We can split the solution $u(x)$ into the sum of a low degree polynomial which satisfies the nonhomogeneous boundary conditions plus a sum over the basis functions $\phi_j(x)$ that satisfy the equivalent homogeneous boundary conditions.

In such a case we proceed as follows:

Setting

$$u(x) = \widetilde{u}(x) + u_e(x), \tag{2.3}$$

where $\widetilde{u}$ is an auxiliary unknown satisfying a modified equation and with homogeneous Robin boundary conditions at both interval extremes, while $u_e(x)$ is an arbitrary function satisfying the original boundary conditions, $a_\pm u_e(\pm 1) + b_\pm D_x u_e(\pm 1) = e_\pm$.