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Optimal control based on the variational iteration method

Samira Berkani, Farida Manseur, Ahmed Maidi *

Département Automatique, Faculté de Génie Electrique et d'Informatique, Université Mouloud MAMMERI, 15 000 Tizi-Ouzou, Algérie

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ABSTRACT

In this study, an approach for designing an optimal control law, based on the variational iteration method, is proposed. The idea consists of formulating the problem of optimal control under a variational problem form by substituting the control variable by its expression derived from the system model in the performance index. Then, the solution of the variational problem obtained is achieved by satisfying the Euler–Lagrange equation, which is in general nonlinear. Therefore, the variational iteration method is adopted to solve the Euler–Lagrange equation obtained, and the optimal control law is easily deduced by performing simple calculations. To illustrate the suggested design approach, a nonlinear system is considered, and the result obtained is compared with the analytical solution to show the effectiveness of the proposed design approach.

1. Introduction

Optimal control problems often arise in system engineering. The objective of optimal control is to find, in open loop control, optimal manipulated variable time profiles for a dynamic system that optimize a given performance index.

The determination of optimal control is a very tedious task, and open-ended due to the nonlinear nature of dynamic systems. Considerable efforts are spent on developing efficient optimal control techniques [1]. Hence, several approaches for solving optimal control problems are suggested in the literature. These can be divided into analytical methods, that were used originally, and numerical methods. The analytical methods include ones based on dynamic programming (Bellman's principle of optimality), Pontryagin's minimum (or maximum) principle and variational calculus, whereas the numerical methods can be grouped into two categories: indirect and direct methods [2].

For analytical methods, dynamic programming (DP) is a method in very general use for treating a bunch of optimization problems. This method is based on the principle of optimality first formulated by Bellman and often used in the analysis and design of automatic control systems [3,4]. Therefore, by differentiating Bellman's function a partial differential equation is obtained, which must satisfy given boundary conditions. Bellman's partial differential equation and the boundary conditions represent necessary conditions for obtaining the minimum of the optimal control problem. Another very efficient approach for solving optimal control problems is by using the Pontryagin minimum principle (PMP) [5,6]. This technique is built on defining the Hamiltonian function by introducing adjoint variables [5]. The necessary condition for having optimal control is optimization of the Hamiltonian function with respect to the control variable [5,6]. Then, the optimal control law is obtained by solving the canonical differential equations, that is, the Hamilton equations that constitute the necessary conditions of optimality according to the minimum principle [7,8]. Dynamic programming and the Pontryagin minimum principle are more general and effective than classic variational calculus. The elementary terms of variational calculus are obtained from Bellman's partial differential equation. The variational calculus [7,9] deals with finding the optimum value of a functional and generally consists in solving the well-known Euler–Lagrange equation [1,8] with imposed boundary conditions.

* Corresponding author.
E-mail address: ahmed.maidi@gmail.com (A. Maidi).

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Optimality conditions for analytical methods are in general not able to provide the optimum since the resulting two-point boundary value problem or Bellman partial differential equation are difficult to solve analytically. Computational methods are therefore needed [2,10,11]. These methods can be split into two categories: the indirect methods (boundary condition iteration and control vector iteration) and the direct methods (the sequential method and the simultaneous method).

In this paper, we address the optimal control of a nonlinear system. The aim is to suggest an optimal control design approach based on the variational iteration method. This method was recently developed by He [12] and has been proved to be reliable, accurate and effective in both the analytic and the numerical solution of differential equations [13]. However, for control problems, the variational iteration method is still not applied. The only contribution in the field has been that of Kucuk [14], where an active optimal control of the Korteweg–de Vries equation that minimizes a quadratic functional is designed, based on the iteration variational method. In contrast to the direct control parameterization used by Kucuk [14], a standard variational calculus method is adopted in this work to achieve nonlinear optimal control. The idea consists of applying the variation iteration method to solve the Euler–Lagrange equation corresponding to the optimal control. Thus, using the variational iteration method, an approximate analytical or numerical solution of the Euler–Lagrange equation can be obtained even if the analytical solution is not available, as is shown by Tatari and Dehghan [15]. The design approach is illustrated by an application example for which an analytical solution exists. The objective is to demonstrate the effectiveness of the proposed approach.

The article is structured as follows. In Section 2, the optimal control problem addressed in this work is formulated. Section 3 is dedicated to the variational problem and its solution based on the Euler–Lagrange equation. Section 4 is devoted to the variational iteration method used to solve differential equations. Section 5 presents the proposed design approach for solving optimal control problems while Section 6 gives an application example that illustrates the proposed design approach. Finally, a conclusion ends the article.

2. The optimal control problem

An optimal control problem is stated as follows. Find the optimal control law $u(t) : [0, t_f] \subset \Re \mapsto \Re$ for minimizing the performance index

$$J = \int_0^{t_f} F(x(t), u(t), t)\, dt \tag{1}$$

subject to

$$\dot{x}(t) = f(x(t), u(t), t) \tag{2}$$

$$x(0) = x_0 \tag{3}$$

and terminal constraints

$$x(t_f) = x_f \tag{4}$$

where $x(t) \in \Re^n$ is the state vector (n is the system order), $u(t) \in \Re$ is the control variable, t_f is the fixed final time, $F : \Re^n \times \Re \times \Re \mapsto \Re$ is a scalar function and continuously differentiable, $f : \Re^n \times \Re \times \Re \mapsto \Re^n$ is a smooth vector field, $x(0)$ is the known initial state and $x(t_f)$ is the fixed final state.

The optimal control is denoted as $u^*(t)$ and it is assumed that the state $x(t)$ and the control $u(t)$ are unconstrained, i.e. there are no limitations (they are unbounded) on the magnitudes of the state and control variable.

There are several approaches that can solve optimal control problems [1]. These can be divided into analytical methods [8], that were used originally, and numerical methods [2,11,16], preferred nowadays. In this work, the calculus of variations [9] that will be discussed below is adopted to solve the formulated optimal control problem.

3. The variational problem

A variational problem is formulated as follows:

$$\hat{J}\big(x^*(t)\big) = \min_{x(t)} \int_0^{t_f} G\big(t, x(t), \dot{x}(t)\big)\, dt \tag{5}$$

or

$$x^*(t) = \underset{x(t)}{\operatorname{argmin}} \int_0^{t_f} G\big(t, x(t), \dot{x}(t)\big)\, dt \tag{6}$$

subject to the following constraints:

$$x(0) = x_0 \tag{7}$$

$$x(t_f) = x_f \tag{8}$$

where $x(t)$ is a differentiable function of a real variable t to be found such that boundary conditions (7) and (8) are satisfied. The solution $x(t)$ is a stationary point of the dynamic optimization problem (5). G is a real-valued function with continuous first partial derivatives.