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The highly accurate block-grid method in solving Laplace's equation for nonanalytic boundary condition with corner singularity

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ABSTRACT

The highly accurate block-grid method for solving Laplace's boundary value problems on polygons is developed for nonanalytic boundary conditions of the first kind. The quadrature approximation of the integral representations of the exact solution around each reentrant corner("singular" part) are combined with the 9-point finite difference equations on the "nonsingular" part. In the integral representations, and in the construction of the sixth order gluing operator, the boundary conditions are taken into account with the help of integrals of Poisson type for a half-plane which are computed with ε accuracy. It is proved that the uniform error of the approximate solution is of order $O(h^6+\varepsilon)$, where h is the mesh step. This estimation is true for the coefficients of singular terms also. The errors of p-order derivatives ($p = 0, 1, \ldots$) in the "singular" parts are $O((h^6 + \varepsilon)r_j^{1/\alpha_j-p})$, r_j is the distance from the current point to the vertex in question and $\alpha_j\pi$ is the value of the interior angle of the jth vertex. Finally, we give the numerical justifications of the obtained theoretical results.

1. Introduction

It is well known that the use of classical finite difference or finite element methods for solving the elliptic boundary value problems with singularities is ineffective. A special construction is usually needed for the numerical scheme near the singularities to get highly accurate approximate solution as given by Andreev [1], Brenner [2], Li and Lu [3], Dosiyev [4], Xenophontos et al. [5], Liu [6] and references therein.

In the last decade, to improve the accuracy of the approximate solution a special emphasis has been placed on the construction of combined methods, in which differential properties of the solution in different parts of the domain are used (see [7–14]).

In [4,10–12] a new combined difference-analytical method, called the block-grid method (BGM), is given for solving the Laplace equation on polygons, when the boundary functions on the sides causing the singular vertices are given as algebraic polynomials of arclength. In BGM, by making an artificial boundary, we reduce the problem to a domain without singularities (the "nonsingular" part of the polygon). Exact boundary conditions used on the artificial boundary are the integral representations around singular vertices, on blocks (the "singular" parts of the polygon). Then, on the "nonsingular" part the Laplace equation is approximated by finite difference equations, and on the "singular" part an exponentially convergent quadrature formula is used. A gluing operator of appropriate order is constructed for gluing together the grids and blocks. The BGM gives a highly accurate approximation not only for the exact solution, but also for its derivatives in the "singular" parts, which is problematic for other methods.

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When the boundary functions are nonanalytic, the integral representations contain additionally the Poisson type integrals for a half plane, which beforehand need an approximation with some $\varepsilon > 0$ accuracy. Hence to construct a high order BGM a special construction of the gluing operator near the boundary of the polygon is needed. Furthermore, from the results in [15] it follows that for the boundary functions from the Hölder classes $C^{6,\lambda}$, $0 < \lambda < 1$, the order of maximum error of 9-point solution in the "nonsingular" part can be $O(h^6)$, where h is the mesh step. Therefore, a construction of the sixth order BGM is reasonable.

In this paper the sixth order block-grid method is constructed and justified for solving the Dirichlet problem for Laplace's equation on staircase polygons i.e., interior angles $\alpha_j\pi$, $\alpha_j = 1/2, 1, 3/2, 2$, with nonanalytic boundary functions from $C^{6,\lambda}$, $0 < \lambda < 1$. To connect the system of equations obtained from the approximation of the integral representations around each singular vertex with the 9-point approximation of the Laplace equation on the "nonsingular" part of the polygon, the sixth order gluing operator given in [4] is developed. In the construction of the gluing operator near the boundaries of the polygon a special representation of the harmonic function through the integrals of Poisson type for a half plane is used. This harmonic function also takes part in the integral representation of the exact solution around each singular vertex. It is proved that the final uniform error is of order $O(h^6 + \varepsilon)$, where h is the mesh step for the "nonsingular" part, and ε is the error of the approximation of the Poisson type integrals. For the errors of p-order derivatives ($p = 0, 1, \ldots$), the difference between the approximate and exact solutions in the "singular" parts (block sectors) is of order $O((h^6 + \varepsilon)r_j^{1/\alpha_j - p})$, where r_j is the distance from the current point to the singular vertex in question. Finally, we illustrate the method of finding a highly accurate solution and its derivatives of the problem in an L-shaped polygon with corner singularity. The error analysis depending on ε, for a fixed ε depending on the mesh size h, and a number of quadrature nodes n are given. The dependence of the results on the smoothness of the boundary functions are also demonstrated. Furthermore, a simple and highly accurate formulae to calculate the coefficients of the singular terms are given.

In [13], the restriction on the boundary functions to be algebraic polynomials on the sides of the polygon causing the singular vertices is also removed. These polynomials are replaced by the functions from the Hölder classes $C^{2,\lambda}$, $0 < \lambda < 1$, and the second order BGM by using linear interpolation gluing operator, is constructed.

2. Integral representation of a solution

Let G be an open simply connected staircase polygon, $\gamma_j, j = 1(1)N$, be its sides, including the ends, enumerated counterclockwise. $\gamma = \gamma_1 \cup \cdots \cup \gamma_N$ be the boundary of G, $\alpha_j\pi$, $\alpha_j \in \{\frac{1}{2}, 1, \frac{3}{2}, 2\}$, be the interior angle formed by the sides γ_{j-1} and γ_j, ($\gamma_0 = \gamma_N$). Denote by $A_j = \gamma_{j-1} \cap \gamma_j$ the vertex of the j-th angle, and by r_j, θ_j a polar system of coordinates with pole in A_j, where the angle θ_j is taken counterclockwise from the side γ_j. Let $C^{k,\lambda}(\Omega)$ be the class of functions that have continuous k-th derivatives on Ω satisfying the Hölder condition with the exponent $\lambda \in (0, 1)$.

We consider the boundary value problem

$$\Delta u = 0 \quad \text{on } G, \qquad u = \varphi_j(s) \quad \text{on } \gamma_j,\ 1 \le j \le N, \tag{1}$$

where $\Delta \equiv \partial^2/\partial x^2 + \partial^2/\partial y^2$, φ_j is a given function on γ_j of arc length s taken along γ, and

$$\varphi_j \in C^{6,\lambda}(\gamma_j). \tag{2}$$

At some vertices A_j, ($s = s_j$) for $\alpha_j = 1/2$ the conjugation conditions

$$\varphi_{j-1}^{(2q)}(s_j) = (-1)^q \varphi_j^{(2q)}(s_j), \quad q = 0, 1, 2, \tag{3}$$

are fulfilled. Let E be the set all j, ($1 \le j \le N$) for which $\alpha_j \ne 1/2$ or $\alpha_j = 1/2$ but (3) is not fulfilled, and let $T_j(r) = \{(r_j, \theta_j) : 0 < r_j < r, 0 < \theta_j < \alpha_j\pi\}$, $j \in E$. In the neighborhood of $A_j, j \in E$ we construct two fixed block-sectors $T_j^i = T_j(r_{ji}) \subset G$, $i = 1, 2$, where $0 < r_{j2} < r_{j1} < \min\{s_{j+1} - s_j, s_j - s_{j-1}\}$.

Let (see [16])

$$\varphi_{j0}(r) = \varphi_j(s_j + r) - \varphi_j(s_j), \qquad \varphi_{j1}(r) = \varphi_{j-1}(s_j - r) - \varphi_{j-1}(s_j),$$

$$Q_j(r_j, \theta_j) = \varphi_j(s_j) + (\varphi_{j-1}(s_j) - \varphi_j(s_j))\theta_j/\alpha_j\pi + \frac{1}{\pi}\sum_{k=0}^{1}\int_0^{\sigma_{jk}} \frac{\widetilde{y}_j \varphi_{jk}(r^{\alpha_j})dr}{(r - (-1)^k\widetilde{x}_j)^2 + \widetilde{y}_j^2}, \tag{4}$$

where

$$\widetilde{x}_j = r_j^{1/\alpha_j}\cos(\theta_j/\alpha_j), \qquad \widetilde{y}_j = r_j^{1/\alpha_j}\sin(\theta_j/\alpha_j), \tag{5}$$

$$\sigma_{jk} = \left|s_{j+1-k} - s_{j-k}\right|^{1/\alpha_j}, \quad j \in E. \tag{6}$$

The function $Q_j(r_j, \theta_j)$ has the following properties:

(i) $Q_j(r_j, \theta_j)$ is harmonic and bounded in the infinite angle $0 < r_j < \infty$, $0 < \theta_j < \alpha_j\pi$;