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Fixed point and convergence theorems for different classes of generalized nonexpansive mappings in CAT(0) spaces

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ABSTRACT

In this paper, we prove the existence of common fixed points in CAT(0) spaces for three different classes of generalized nonexpansive mappings including a quasi-nonexpansive single valued mapping, a pointwise asymptotically nonexpansive mapping, and a multivalued mapping satisfying the conditions (E) and (C_λ) for some $\lambda \in (0, 1)$. Moreover, we introduce an iterative process for these mappings and prove $\triangle$-convergence and strong convergence theorems for such an iterative process in CAT(0) spaces.

1. Introduction

Let (X, d) be a metric space. A mapping $T : X \to X$ is called

(i) nonexpansive if $d(Tx, Ty) \le d(x, y)$ for all $x, y \in X$,
(ii) quasi-nonexpansive if the set $F(T)$ of fixed points of T is nonempty and $d(Tx, Ty) \le d(x, y)$ for all $x \in X$ and $y \in F(T)$,
(iii) pointwise asymptotically nonexpansive if there exists a sequence of functions $\alpha_n(x) \ge 1$ with $\lim_{n\to\infty} \alpha_n(x) = 1$ such that

$$d(T^n(x), T^n(y)) \le \alpha_n(x) d(x, y), \quad n \ge 1,\ x, y \in X.$$

(iv) In case when each α_n is constant, T is called asymptotically nonexpansive.

The class of pointwise asymptotically nonexpansive mappings was introduced by Kirk and Xu [1] as a generalization of the class of asymptotically nonexpansive mappings which had already been introduced by Goebel and Kirk in [2]. It is immediately clear that a nonexpansive mapping is pointwise asymptotically nonexpansive.

In [3], Garcia-Falset et al. introduced two types of generalization for nonexpansive mappings.

Definition 1.1. Let (X, d) be a metric space and $\mu \ge 1$. A mapping $T : X \to X$ is said to satisfy condition (E_μ) if

$$d(x, Ty) \le \mu\, d(x, Tx) + d(x, y), \quad x, y \in X.$$

We say that T satisfies condition (E) whenever T satisfies the condition (E_μ) for some $\mu \ge 1$.

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Definition 1.2. Let (X, d) be a metric space and $\lambda \in (0, 1)$. A mapping $T : X \to X$ is said to satisfy condition (C_λ) if

$$\lambda\, d(x, Tx) \le d(x, y) \Longrightarrow d(Tx, Ty) \le d(x, y), \quad x, y \in X.$$

Very recently, the current authors have modified these conditions to incorporate the multivalued mappings, and proved some fixed point theorems for multivalued mappings satisfying these conditions in CAT(0) spaces [4]. In this paper, we consider a CAT(0) space, and intend to prove the existence of common fixed points for three different classes of generalized nonexpansive mappings including a quasi-nonexpansive single valued mapping, a pointwise asymptotically nonexpansive mapping, and a multivalued mapping satisfying the condition (E) and (C_λ) for some $\lambda \in (0, 1)$. Moreover, we introduce an iterative process for these mappings and prove $\triangle$-convergence and strong convergence theorems for such an iterative process in CAT(0) spaces. Our result generalizes a number of recent known results; including that of Abkar and Eslamian [4], Hussain and Khamsi [5], Khan and Abbas [6], and of Dhompongsa and Panyanak [7].

2. Preliminaries

Let (X, d) be a metric space. A geodesic path joining $x \in X$ and $y \in X$ is a map c from a closed interval $[0, r] \subset \mathbb{R}$ to X such that $c(0) = x$, $c(r) = y$ and $d(c(t), c(s)) = |t - s|$ for all $s, t \in [0, r]$. In particular, the mapping c is an isometry and $d(x, y) = r$. The image of c is called a geodesic segment joining x and y which when unique is denoted by $[x, y]$. For any $x, y \in X$, we denote the point $z \in [x, y]$ such that $d(x, z) = \alpha d(x, y)$ by $z = (1 - \alpha)x \oplus \alpha y$, where $0 \le \alpha \le 1$. The space (X, d) is called a geodesic space if any two points of X are joined by a geodesic, and X is said to be uniquely geodesic if there is exactly one geodesic joining x and y for each $x, y \in X$. A subset D of X is called convex if D includes every geodesic segment joining any two points of itself.

A geodesic triangle $\triangle(x_1, x_2, x_3)$ in a geodesic metric space (X, d) consists of three points in X (the vertices of $\triangle$) and a geodesic segment between each pair of points (the edges of $\triangle$). A comparison triangle for $\triangle(x_1, x_2, x_3)$ in (X, d) is a triangle $\overline{\triangle}(x_1, x_2, x_3) := \triangle(\overline{x_1}, \overline{x_2}, \overline{x_3})$ in the Euclidean plane $\mathbb{R}^2$ such that $d_{\mathbb{R}^2}(\overline{x_i}, \overline{x_j}) = d(x_i, x_j)$ for $i, j \in \{1, 2, 3\}$.

A geodesic metric space X is called a CAT(0) space if all geodesic triangles of appropriate size satisfy the following comparison axiom.

Let $\triangle$ be a geodesic triangle in X and let $\overline{\triangle}$ be its comparison triangle in $\mathbb{R}^2$. Then $\triangle$ is said to satisfy the CAT(0) inequality if for all $x, y \in \triangle$ and all comparison points $\bar{x}, \bar{y} \in \overline{\triangle}$, $d(x, y) \le d_{R^2}(\bar{x}, \bar{y})$.

The following properties of a CAT(0) space are useful (see [8]):

(i) A CAT(0) space X is uniquely geodesic;
(ii) For any $x \in X$ and any closed convex subset $D \subset X$, there is a unique closest point to x.

Let $\{x_n\}$ be a bounded sequence in X and D be a nonempty bounded subset of X. We associate this sequence with the number

$$r = r(D, \{x_n\}) = \inf\{r(x, \{x_n\}) : x \in D\},$$

where

$$r(x, \{x_n\}) = \limsup_{n\to\infty} d(x_n, x),$$

and the set

$$A = A(D, \{x_n\}) = \{x \in D : r(x, \{x_n\}) = r\}.$$

The number r is known as the *asymptotic radius* of $\{x_n\}$ relative to D. Similarly, the set A is called the *asymptotic center* of $\{x_n\}$ relative to D. In a CAT(0) space, the asymptotic center $A = A(D, \{x_n\})$ of $\{x_n\}$ consists of exactly one point when D is closed and convex. A sequence $\{x_n\}$ in a CAT(0) space X is said to be $\triangle$-convergent to $x \in X$ if x is the unique asymptotic center of every subsequence of $\{x_n\}$. Notice that given $\{x_n\} \subset X$ such that $\{x_n\}$ is $\triangle$-convergent to x and given $y \in X$ with $x \ne y$,

$$\limsup_{n\to\infty} d(x, x_n) < \limsup_{n\to\infty} d(y, x_n).$$

Thus every CAT(0) space X satisfies the Opial property.

Lemma 2.1 (*[9]*). *Every bounded sequence in a complete CAT(0) space has a $\triangle$-convergent subsequence.*

Lemma 2.2 (*[10]*). *If D is a closed convex subset of a complete CAT(0) space and if $\{x_n\}$ is a bounded sequence in D, then the asymptotic center of $\{x_n\}$ is in D.*

Lemma 2.3 (*[7]*). *If $\{x_n\}$ is a bounded sequence in a complete CAT(0) space X with $A(\{x_n\}) = \{x\}$, and $\{u_n\}$ is a subsequence of $\{x_n\}$ with $A(\{u_n\}) = \{u\}$, and the sequence $\{d(x_n, u)\}$ converges, then $x = u$.*

Theorem 2.4 (*[5]*). *Let D be a nonempty closed convex subset of a complete CAT(0) space X. Suppose $f : D \to D$ is a pointwise asymptotic nonexpansive mapping. If $\{x_n\}$ is a sequence in D such that $\lim_{n\to\infty} d(fx_n, x_n) = 0$ and $\triangle - \lim_n x_n = v$. Then $v = f(v)$.*