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Hadamard-type and Bullen-type inequalities for Lipschitzian functions and their applications

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ABSTRACT

In this paper, we shall establish some Hadamard-type and Bullen-type inequalities for Lipschitzian functions and give several applications for special means.

1. Introduction

Throughout this paper, let $L \geq 0$ and $a < b$ in $\mathbb{R}$.

The inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{f(a)+f(b)}{2} \tag{1.1}$$

which holds for all convex functions $f : [a, b] \to \mathbb{R}$, is known in the literature as Hadamard's inequality [1].

See [2–14], the results of which are the generalization, improvement and extension of the famous integral inequality (1.1).

Recently, Tseng et al. [9] have established the following Hadamard-type inequality which refines the inequality (1.1).

Theorem A. *Suppose that $f : [a, b] \to \mathbb{R}$ is a convex function on $[a, b]$. Then we have the inequalities:*

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &\leq \frac{1}{2}\left[f\left(\frac{3a+b}{4}\right) + f\left(\frac{a+3b}{4}\right)\right] \\ &\leq \frac{1}{b-a}\int_a^b f(x)dx \\ &\leq \frac{1}{2}\left[f\left(\frac{a+b}{2}\right) + \frac{f(a)+f(b)}{2}\right] \leq \frac{f(a)+f(b)}{2}. \end{aligned} \tag{1.2}$$

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The third inequality in (1.2) is known in the literature as Bullen's inequality.

In what follows we recall the following definition.

Definition 1. A function $f : I \subset \mathbb{R} \to \mathbb{R}$ is called an L-Lipschitzian function on the interval I of real numbers if

$$|f(x) - f(y)| \leq L|x - y|$$

for all $x, y \in I$.

Dragomir et al. [5] and Matić and Pečarić [8] established the following Hadamard-type inequalities for Lipschitzian functions.

Theorem B. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be an L-Lipschitzian function on the interval I of real numbers and $a, b \in I$. Then, we have the following inequalities*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{L(b-a)}{4} \tag{1.3}$$

and

$$\left|f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \leq \frac{L(b-a)}{4}. \tag{1.4}$$

In this paper, we shall establish some Hadamard-type and Bullen-type inequalities for Lipschitzian functions and give several applications for special means.

2. Hadamard-type inequalities for Lipschitzian functions

Throughout this section, let I be an interval in $\mathbb{R}$, $a \leq A \leq B \leq b$ in I and let $f : I \to \mathbb{R}$ be an L-Lipschitzian function.

In the next theorem, let $\alpha \in [0, 1]$, $V = (1-\alpha)a + \alpha b$, and define V_α as follows:

(1) If $a \leq V \leq A \leq B \leq b$, then

$$V_\alpha(A, B) = (A-a)^2 - (A-V)^2 + (B-V)^2 + (b-B)^2.$$

(2) If $a \leq A \leq V \leq B \leq b$, then

$$V_\alpha(A, B) = (A-a)^2 + (V-A)^2 + (B-V)^2 + (b-B)^2.$$

(3) If $a \leq A \leq B \leq V \leq b$, then

$$V_\alpha(A, B) = (A-a)^2 + (V-A)^2 + (b-B)^2 - (V-B)^2.$$

Theorem 1. *Let $A, B, \alpha, V, V_\alpha$ and the function f be defined as above. Then we have the inequality*

$$\left|\alpha f(A) + (1-\alpha)f(B) - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{LV_\alpha(A, B)}{2(b-a)}. \tag{2.1}$$

Proof. Using the hypothesis of f, we have the following inequality

$$\begin{aligned}\left|\alpha f(A) + (1-\alpha)f(B) - \frac{1}{b-a}\int_a^b f(x)dx\right| &= \frac{1}{b-a}\left|\int_a^V [f(A) - f(x)]\,dx + \int_V^b [f(B) - f(x)]\,dx\right| \\ &\leq \frac{1}{b-a}\left[\int_a^V |f(A) - f(x)|\,dx + \int_V^b |f(B) - f(x)|\,dx\right] \\ &\leq \frac{L}{b-a}\left[\int_a^V |A - x|\,dx + \int_V^b |B - x|\,dx\right].\end{aligned} \tag{2.2}$$

Now, using simple calculations, we obtain the following identities $\int_a^V |A-x|\,dx$ and $\int_V^b |B-x|\,dx$.

(1) If $a \leq V \leq A \leq B \leq b$, then we have

$$\int_a^V |A-x|\,dx = \frac{(A-a)^2 - (A-V)^2}{2} \quad \text{and} \quad \int_V^b |B-x|\,dx = \frac{(B-V)^2 + (b-B)^2}{2}.$$