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Pricing a contingent claim with random interval or fuzzy random payoff in one-period setting

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ABSTRACT

This article proposes a method for pricing a contingent claim with random interval and fuzzy random payoff. On introduction of the acceptability concept based on classical no-arbitrage argument, a price interval and a fuzzy price are obtained in random interval market and fuzzy random market, respectively. New definitions on replicative strategies, sub-replicative and sup-replicative ones, in two market setting are given. Some interesting results similar to those in the classical random market are presented.

1. Introduction

In modern financial theories, prices of uncertain assets are modeled as random variables or stochastic processes. With the help of probability theory, we have seen great advances in financial analysis, including asset pricing, portfolio selection, risk management and consumption optimization, etc. If we model an uncertain quantity as a random variable, we should know all possible realizations of the quantity and the probability of the occurrence of every possible realization.

However, there are many settings where we can't model as random variables. For example [1], we do know the market is either "bullish" or "bearish", but we can't give precise values in two states. We should use other uncertain tools to target this imprecision. According to different properties of an uncertain quantity, it can be modeled as a random variable, an interval number, a fuzzy number or a fuzzy random variable [2]. In the example above, it seems more confident to view stock prices as two interval or fuzzy numbers. Fuzzy theory can be used to handle cases where vague or ambiguous information, such as "the price is about $20" and "the price will have a big jump", are involved.

With the exception of randomness in finance, uncertainties have motivated research in portfolio selection, such as [3–6] and asset pricing, such as [1,7–9]. We can see that different uncertain tools are used to model the security. A single-period setting model for contingent claim pricing is the simplest model in pricing theory[10–12]. Many important results and methods for pricing are extended from single period setting. All discussions are based on algebra and probability theory. This article will extend classical results to the market where all securities have random interval or fuzzy random payoffs.

This paper is organized as follows. In Section 2, we recall classical results and methods on contingent claim pricing theory in one-period setting. In Section 3, we will introduce the market with random interval payoffs. After proposing a concept

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of acceptable market based on no-arbitrage principle, we give the price interval for contingent claim with random interval payoffs. In Section 4, we will discuss the pricing problem of a contingent claim with fuzzy random payoffs. A procedure and an example will be given to explain the method. Some discussion and further development are finally given in Section 5.

2. Overview of classical contingent claim pricing theory in one-period setting

Throughout this article, we denote $Y = (y_1, \dots, y_n) \geq 0$ for all $y_i \geq 0, i = 1, 2, \dots, n$; $Y > 0$ for $Y \geq 0$ and $Y \neq 0$; $Y \gg 0$ for all $y_i > 0, i = 1, 2, \dots, n$.

In one-period setting, there are two trading dates: date $t = 0$ and date $t = 1$. At $t = 0$, every security has its deterministic price quoted in the market. At $t = 1$, suppose there are M possible states, denoted as $\Omega = \{w_1, \dots, w_M\}$. There are N basic securities (including stocks and bonds), whose prices at $t = 0$ are given as $S = (S_1, S_2, \dots, S_N)^{\mathrm{T}}$. At date $t = 1$, any basic securities have random payoffs with state sets Ω. Because of the finiteness of possible state set, N basic securities have a payoff matrix $D = [D_{ij}]_{N\times M}$, where D_{ij} is the payoff of the ith security at state w_j. Denote the market composed of above N basic securities be $\mathcal{M} = (S, D)$.

$\theta = (\theta_1, \dots, \theta_N)^{\mathrm{T}}$ is called a portfolio (trading strategy) of N basic securities, where θ_i is the unit amount of the ith security. As $\theta_i < 0$, it means the investor sells $-\theta_i$ units of the ith security short. Here we don't add any constraints on short-selling. Portfolio θ has its value $S^{\mathrm{T}}\theta$ at $t = 0$, and payoff vector $D^{\mathrm{T}}\theta$ at $t = 1$.

Definition 2.1. θ is called an arbitrage, if $S^{\mathrm{T}}\theta < 0, D^{\mathrm{T}}\theta \geq 0$ or $S^{\mathrm{T}}\theta \leq 0, D^{\mathrm{T}}\theta > 0$ hold at the same time.

Definition 2.2. A vector $\psi \gg 0$ is called a state price vector for the market $\mathcal{M}$, if $S = D\psi$.

Lemma ([12]). *$\mathcal{M}$ has no arbitrage opportunity if and only if the market has at least one state price vector.*

Now introduce a contingent claim whose price, h, at date 0 is to be determined with payoff given by $X = (X_1, \dots, X_M)^{\mathrm{T}}$. We want to get the price for the claim X.

The price(price interval) for X will be got by no-arbitrage argument, which means the price h should be determined such that the introduction of X will not lead to arbitrage opportunities. h can be got from two ways: the first from state price vectors, and the second from replicative strategies.

Denote $\Psi = \{\psi \gg 0 | S = D\psi\}$ be all state price vectors in the market $\mathcal{M}$. From the lemma, we get $\Psi \neq \emptyset$ under no-arbitrage condition.

Definition 2.3. The market $\mathcal{M}$ is complete if any contingent claim X can be replicated by basic securities.

A complete market is an idealized market. In practice, we can't get any complete markets. Following theorem gives the relation between completeness and state price vectors.

Theorem 2.1. *$\mathcal{M}$ is complete, if and only if there is a unique state price vector.*

Set $h_- = \inf_{\psi\in\Psi} X^{\mathrm{T}}\psi,\ h_+ = \sup_{\psi\in\Psi} X^{\mathrm{T}}\psi$. Then the arbitrage-free price interval is $[h_-, h_+]$.

Proposition 2.1 ([11]). *Under no-arbitrage principle,*

(1) *if $h_- = h_+$, the claim X has a unique price given by h_- or h_+;*
(2) *if $h_- < h_+$, X has a price interval (h_-, h_+) which has two properties:*
   (a) *at any price level in the open interval, there exists no arbitrage opportunity;*
   (b) *at any price level out of the closed interval $[h_-, h_+]$, arbitrage opportunities exist.*

From above proposition, in a complete market, any contingent claim has a determined price. While in an incomplete market, only those replicable claims have unique prices. From no-arbitrage principle, claims that can't be replicated have a price interval with the property in above proposition.

Definition 2.4. (1) If θ satisfies $D^{\mathrm{T}}\theta \geq X$, we call θ a sup-replicative strategy for X. Denote all sup-replicative strategies for X be $\Theta_p(X) = \{\theta | D^{\mathrm{T}}\theta \geq X, \theta \in \mathbb{R}^N\}$.

(2) If θ satisfies $D^{\mathrm{T}}\theta \leq X$, we call θ a sub-replicative strategy for X. Denote all sub-replicative strategies for X be $\Theta_b(X) = \{\theta | D^{\mathrm{T}}\theta \leq X, \theta \in \mathbb{R}^N\}$.

By duality of linear programming for h_- and h_+, we can get following explanation for the price interval.

Proposition 2.2 ([11]).

$$h_- = \max_{\theta\in\Theta_b(X)} S^{\mathrm{T}}\theta, \qquad h_+ = \min_{\theta\in\Theta_p(X)} S^{\mathrm{T}}\theta.$$

The proposition above tells us, the minimum feasible price for X equals to the maximal value of sub-replicative portfolio of X; and the maximum feasible price for X equals to the minimal value of sup-replicative portfolio of X.