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Contiguous relations and their computations for ${}_2F_1$ hypergeometric series

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ABSTRACT

The hypergeometric function ${}_2F_1[a_1, a_2; a_3; z]$ plays an important role in mathematical analysis and its application. Gauss defined two hypergeometric functions to be contiguous if they have the same power-series variable, if two of the parameters are pairwise equal, and if the third pair differs by ± 1. He showed that a hypergeometric function and any two other contiguous to it are linearly related. In this paper, we present an interesting formula as a linear relation of three shifted Gauss polynomials in the three parameters a_1, a_2 and a_3. More precisely, we obtained a recurrence relation including

$${}_2F_1[a_1 + \alpha_1, a_2; a_3; z], \qquad {}_2F_1[a_1, a_2 + \alpha_2; a_3; z] \quad \text{and} \quad {}_2F_1[a_1, a_2; a_3 + \alpha_3; z]$$

for any arbitrary integers α_1, α_2 and α_3.

1. Introduction

The hypergeometric function ${}_2F_1[a_1, a_2; a_3; z]$ plays an important role in mathematical analysis and its application. This function allows us to solve many interesting problems, such as conformal mapping of triangular domains bounded by line segments or circular arcs, various problems of quantum mechanics.

Almost all of the elementary functions of mathematics are either hypergeometric or ratios of hypergeometric functions. Many of the non elementary functions that arise in mathematics and physics also have representations as hypergeometric series.

There are three very important approaches to hypergeometric functions. First, Euler's fractional integral representations leads easily to the derivation of essential identities and transformations of hypergeometric functions. A second order linear differential equation satisfied by a hypergeometric function provides a second method. This equation was also found by Euler and then studied by Gauss. Still later, Riemann observed that a characterization of second-order equations with three regular singularities gives a powerful technique, involving minimal calculation, for obtaining formulas for hypergeometric functions. Third, Barnes expressed a hypergeometric function as a contour integral, which can be seen as a Meline inversion formula. Some integrals that arise here are really extensions of beta integrals. They also appear in the orthogonality relations for some special orthogonal polynomials.

The major development of the theory of the hypergeometric function was carried out by Gauss and published in his famous memoir of 1812. Some important results concerning the hypergeometric function had been developed by Euler and others, but it was Gauss who made the first systematic study of the series that defines this function.

In (1812), Gauss presented to the Royal Society of Sciences at Göttingen his famous paper (Gauss (1813)) in which he considered the infinite series

$$1 + \frac{a_1 a_2}{1.a_3} z + \frac{a_1(a_1+1)a_2(a_2+1)}{1.2.a_3(a_3+1)} z^2 + \frac{a_1(a_1+1)(a_1+2)a_2(a_2+1)(a_2+2)}{1.2.3.a_3(a_3+1)(a_3+2)} z^3 + \cdots \tag{1.1}$$

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as a function of a_1, a_2, a_3, z, where it is assumed that $a_3 \neq 0, -1, -2, \ldots$, so that no zero factors appear in the denominators of the terms of the series. He showed that the series converges absolutely for $|z| < 1$, and for $|z| = 1$ when $Re(a_3-a_1-a_2) > 0$, gave its (contiguous) recurrence relations, and derived his famous formula

$$F(a_1, a_2; a_3; 1) = \frac{\Gamma(a_3)\Gamma(a_3 - a_1 - a_2)}{\Gamma(a_3 - a_1)\Gamma(a_3 - a_2)}, \qquad Re(a_3 - a_1 - a_2) > 0 \tag{1.2}$$

for the sum of his series when $z = 1$ and $Re(a_3 - a_1 - a_2) > 0$.

Although Gauss used the notation $F(a_1, a_2, a_3, z)$ for his series, it is now customary to use $F[a_1, a_2; a_3; z]$ or either of the notations

$${}_2F_1(a_1, a_2; a_3; z), \qquad {}_2F_1\left[\begin{matrix} a_1, a_2 \\ a_3 \end{matrix}; z\right]$$

for the series (and for its sum when it converges), because these notations separate the numerator parameters a_1, a_2 from the denominator parameter a_3 and the variable z. For more details about hypergeometric series and their contiguous relations, see [1–5].

Gauss defined two hypergeometric functions to be contiguous if they have the same power-series variable, if two of the parameters are pairwise equal, and if the third pair differs by ± 1. He showed that a hypergeometric function and any two other contiguous to it are linearly related. Since there are six contiguous to a given ${}_2F_1$, one get a total of 15 relations. In fact, only four of the fifteen are really independent, as all others may be obtained by elimination and use of the fact that the ${}_2F_1$ is symmetric in a_1 and a_2.

Applications of contiguous relations range from the evaluation of hypergeometric series to the derivation of summation and transformation formulas for such series. They can be used to evaluate a hypergeometric function that is contiguous to a hypergeometric series which can be evaluated. Contiguous relations are also used to imply some continued fraction expansions for hypergeometric functions as well as to make a correspondence between Lie algebras and special functions, such correspondence yields formulas of special functions [6].

In [7], several properties of coefficients of these general contiguous relations were proved and then used to propose effective ways to compute contiguous relations. In [8], contiguous relations were used to establish and prove sharp inequalities between the Gaussian hypergeometric function and the power mean. These results extend known inequalities involving the complete elliptic integral and the hypergeometric mean. In [9], some consequences of the contiguous relations of ${}_2F_1$ were proved, while in [10], a new method of the shifted operators for computing the contiguous relations of ${}_2F_1$ was introduced. More details about contiguous relations and their application can be found in [11–16].

This paper is organized as follows, in the next section, we recall the definition of shifted operators and mentioned some helpful results concerning the recurrence relations of Gauss functions, while in Section 3, we present our main results and give the proofs of our main theorems. Finally, in Section 4, with the help of the computer algebra system *Mathematica*, two computational examples using the results obtained in Section 3 are presented.

2. Preliminaries and tools

Definition 2.1. Let $\mathcal{A}_i^{\alpha_i} : X \to X$, $(i = 1, 2, 3)$, where X is the set of all Gauss' functions ${}_2F_1[a_1, a_2; a_3; z]$ with variable z, and parameters a_1, a_2 and a_3 such that $a_3 \neq 0, -1, -2, \ldots$, then

$$\mathcal{A}_1^{\alpha_1}\left(C[a_1, a_2, a_3]\,{}_2F_1[a_1, a_2; a_3; z]\right) = C[a_1 + \alpha_1, a_2, a_3]\,{}_2F_1[a_1 + \alpha_1, a_2; a_3; z]$$
$$\mathcal{A}_2^{\alpha_2}\left(C[a_1, a_2, a_3]\,{}_2F_1[a_1, a_2; a_3; z]\right) = C[a_1, a_2 + \alpha_2, a_3]\,{}_2F_1[a_1, a_2 + \alpha_2; a_3; z]$$
$$\mathcal{A}_3^{\alpha_3}\left(C[a_1, a_2, a_3]\,{}_2F_1[a_1, a_2; a_3; z]\right) = C[a_1, a_2, a_3 + \alpha_3]\,{}_2F_1[a_1, a_2; a_3 + \alpha_3; z]$$

where α_i, $i = 1, 2, 3$ are any integers, and $C[a_1, a_2, a_3]$ is an arbitrary constant function of a_1, a_2 and a_3 such that for any such operators

$$\mathcal{A}_i^{\alpha_i}\mathcal{A}_i^{-\alpha_i}\left(C[a_1, a_2, a_3]\,{}_2F_1[a_1, a_2; a_3; z]\right) = \mathcal{I}\left(C[a_1, a_2, a_3]\,{}_2F_1[a_1, a_2; a_3; z]\right)$$

and $\mathcal{I}$ is the identity operator defined on X with

$$\begin{aligned}\mathcal{I}^k\left(C[a_1, a_2, a_3]\,{}_2F_1[a_1, a_2; a_3; z]\right) &= \mathcal{I}\left(C[a_1, a_2, a_3]\,{}_2F_1[a_1, a_2; a_3; z]\right) \\ &= C[a_1, a_2, a_3]\,{}_2F_1[a_1, a_2; a_3; z]; \quad \forall F \in X.\end{aligned}$$

The five shifted operators of the 1st degree $\mathcal{A}_3^{-1}$, $\mathcal{A}_2^{-1}$, $\mathcal{A}_1^{-1}$, $\mathcal{A}_2$ and $\mathcal{A}_3$ in terms of the two operators $\mathcal{A}_1$ and $\mathcal{I}$ were introduced in [10, Theorem (2)], as follows:

$$\mathcal{A}_3^{-1} = \frac{a_1}{a_3 - 1}\mathcal{A}_1 + \frac{a_3 - a_1 - 1}{a_3 - 1}\mathcal{I}; \quad a_3 \neq 1 \tag{2.1}$$