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Guaranteed and sharp a posteriori error estimates in isogeometric analysis

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ABSTRACT

We present functional-type a posteriori error estimates in isogeometric analysis. These estimates, derived on functional grounds, provide guaranteed and sharp upper bounds of the exact error in the energy norm. Moreover, since these estimates do not contain any unknown/generic constants, they are fully computable, and thus provide quantitative information on the error. By exploiting the properties of non-uniform rational B-splines, we present efficient computation of these error estimates. The numerical realization and the quality of the computed error distribution are addressed. The potential and the limitations of the proposed approach are illustrated using several computational examples.

1. Introduction

The geometry representations in finite element methods (FEM) and computer aided design (CAD) have been developed independent of each other, and are optimized for the purposes within their respective fields. As a consequence, the representations are different from each other, and a transfer of geometry information from CAD to FEM programmes (and vice versa) requires a transformation of geometry data. These transformations are, in general, not only costly, but also prone to approximation errors, and may require manual input.

Isogeometric analysis (IGA), introduced by Hughes et al. [1], see also [2], aims at closing this gap between FEM and CAD. The key observation is that it is a widespread standard in CAD to use geometry representations based on non-uniform rational B-splines (NURBS), and that these NURBS basis functions have properties which make them suitable as basis functions for FEM. Instead of transforming the geometry data to a conventional FEM representation, the original geometry description is used directly, and the underlying NURBS functions are used as basis for the discrete solution. This way, the geometry is represented *exactly* in the sense that the geometry obtained from CAD is not changed. Thus, the need for data transformation is eliminated, and furthermore, the exact representation from the coarsest mesh is preserved throughout the refinement process. IGA has been thoroughly studied and analyzed (see, e.g., [3–7]), and its potential has been shown by successful applications to a wide range of problems (see, e.g., [8–12]).

As mentioned above, the most widely used spline representations in CAD are based on NURBS. The straightforward definition of NURBS basis functions leads to a tensor-product structure of the basis functions, and thus of the discretization. Since naive mesh refinement in a tensor-product setting has global effects, the development of local refinement strategies for isogeometric analysis is a subject of current active research. Such local refinement techniques include, for example,

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T-splines [13–17], truncated hierarchical B-splines (THB-splines) [18,19], polynomial splines over hierarchical T-meshes (PHT-splines) [20,21], and locally-refinable splines (LR-splines) [22,23].

The issue of adaptive, local refinement is closely linked to the question of efficient a posteriori error estimation (see, e.g., [24,25] for a general overview on error estimators). In the light of adaptive refinement, an error estimator has to identify the areas where further refinement is needed due to the local error being significantly larger than in the rest of the domain. Hence, an accurate indication of the error distribution is essential. Another important objective in computing a posteriori error estimates is to address the *quality assurance*, i.e., to quantify the error in the computed solution with certain degree of *guarantee*. However, a posteriori error estimation in isogeometric analysis is still in an infancy stage. To the best of the authors' knowledge, the only published results are [26–31,21,32,33].

A posteriori error estimates based on hierarchical bases, proposed by Bank and Smith [34], have been used in [27,31]. The reliability and efficiency of this approach is subjected to the saturation assumption on the (enlarged) underlying space and the constants in the strengthened Cauchy inequality. As the authors remarked, the first assumption is critical and its validity depends on the considered example. Moreover, an accurate estimation of constants in the strengthened Cauchy inequality requires the solution of generalized minimum eigenvalue problem. As noted in [28, p. 41], this approach delivers *less than satisfactory* results.

Residual-based a posteriori error estimates have been used in [28,21,32,33]. This approach requires the computation of constants in Clement-type interpolation operators. Such constants are mesh (element) dependent, often generic/unknown or incomputable for general element shape; and the global constant often over-estimates the local constants, and thus the exact error. This fact has been explicitly stated by the authors in [28, pp. 42–43] and in [21, Remark 1].

Goal-oriented error estimation approach has been studied in [26,29,30]. The results presented in these studies show that neither the estimates of this approach are *guaranteed* to be an upper bound, nor the efficiency indices of the estimates are sharp. Moreover, this approach also requires the solution of an adjoint problem, the cost of which cannot be entirely neglected.

The approach of Zienkiewicz–Zhu type a posteriori error estimates is based on post-processing of approximate solutions, and depend on the superconvergence properties of the underlying basis. To the best of authors' knowledge, superconvergence properties for B-splines (NURBS) functions are not yet known.

Summarily, in general situations, the reliability and efficiency of these methods often depend on undetermined constants, which is not suitable for quality assurance purposes. In this paper, we present *functional-type a posteriori error estimates* for isogeometric discretizations. These error estimates, which were introduced in [35–37] and have been studied for various fields (see [25] and the references therein), provide guaranteed, sharp and fully computable bounds (without any generic undetermined constants). These estimates are derived on purely functional grounds (based on integral identities or functional analysis) and are thus applicable to any conforming approximation in the respective space. For elliptic problems with the weak solution $u \in H_0^1(\Omega)$, these error bounds involve computing an auxiliary function $y \in H(\Omega, \text{div})$. In order to get a sharp estimate, this function y is computed by solving a *global* problem. This could be perceived as a drawback when compared to error estimation techniques which rely on local computations and are thus apparently cheaper. However, as briefly explained above, our emphasis is not only on adaptivity, but also on *quantifying the error in the computed solution* (and thus guaranteeing the quality of the computed solution). Therefore, the associated cost should be weighed against the stated objectives. To the best of authors' knowledge, there is no other, particularly cheaper, method available which can fulfill these objectives in general situations. In this paper, we will elaborate how such estimates can be computed efficiently by a proper set-up of the global problem.

Two aspects motivate the application of functional-type error estimates in IGA. Firstly, unlike the standard Lagrange basis functions, NURBS basis functions of degree p are, in general, globally C^{p-1}-continuous. Hence, NURBS basis functions of degree $p \geq 2$ are, in general, at least C^1-continuous, and therefore, their gradients are automatically in $H(\Omega, \text{div})$. Thereby, we avoid constructing complicated functions in $H(\Omega, \text{div})$, in particular for higher degrees (see, e.g., [38–40]). Secondly, since the considered problem is solved in an isogeometric setting, an efficient implementation of NURBS basis functions is readily available, which can be used to construct the above mentioned function y. Hence, applying the technique of functional-type a posteriori error estimation in a setting that relies only on the use of already available NURBS basis functions is greatly appealing.

The remainder of this paper is organized as follows. In Section 2, we define the model problem, and recall the definition and some important properties of B-spline and NURBS basis functions. In Section 3, we first recall functional-type a posteriori error estimates and known implementation issues. Then, we derive a quality criterion and the local error indicator. In Section 4, we discuss a cost-efficient realization of the proposed error estimator using an illustrative numerical example. Further numerical examples are presented in Section 5, and finally, conclusions are drawn in Section 6.

2. Preliminaries

In order to fix notation and to provide an overview, we define the model problem and recall the definition and some aspects of isogeometric analysis in this section.