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The method of approximate particular solutions for the time-fractional diffusion equation with a non-local boundary condition

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ABSTRACT

In this paper, we consider the numerical solution of the time-fractional diffusion equation with a non-local boundary condition. The method of approximate particular solutions (MAPS) using multiquadric radial basis function (MQ-RBF) is employed for this equation. Due to the accuracy of the MQ-based meshless methods is severely influenced by the shape parameter, we adopt a leave-one-out cross validation (LOOCV) algorithm proposed by Rippa [34] to enhance the performance of the MAPS. The numerical results obtained show that the proposed numerical algorithm is accurate and computationally efficient for solving time-fractional diffusion equation with a non-local boundary condition.

1. Introduction

Fractional calculus has attracted considerable attention during the past several decades due to its widespread applications in diverse fields of science and engineering [1]. Because of the non-local properties of fractional operators, obtaining the analytical solutions of the fractional differential equations (FDEs) is more challenging or sometimes even impossible. Hence the proposal, development, and analysis of numerical methods to solve FDEs are at present a quite active field of research, and many methods have been considered, for instance, finite difference method [2–4], finite element method [5,6], spectral method [7–9], meshless method [10–14], and so on.

In the current work, a numerical investigation would be given to approximate the solution of the following two-dimensional time-fractional diffusion equation (TFDE):

$$ {}^{c}D_t^{\alpha} u(\mathbf{x}, t) = \Delta u(\mathbf{x}, t) + f(\mathbf{x}, t), \quad \mathbf{x} \in \Omega,\ t \in (0, T], \tag{1}$$

with the following initial and boundary conditions:

$$u(\mathbf{x}, 0) = u_0(\mathbf{x}), \quad \mathbf{x} \in \Omega, \tag{2}$$
$$u(\mathbf{x}, t) = g_1(\mathbf{x}, t), \quad \mathbf{x} \in \Gamma_1,\ t \in (0, T], \tag{3}$$
$$u(\mathbf{x}, t) = g_2(\mathbf{x})h(t), \quad \mathbf{x} \in \Gamma_2,\ t \in (0, T], \tag{4}$$

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and the non-local integral condition:

$$\int_{\Omega} u(\mathbf{x}, t)d\mathbf{x} = m(t), \quad t \in (0, T], \tag{5}$$

where $u(\mathbf{x}, t)$ and $h(t)$ are unknown functions, $u_0(\mathbf{x})$, $g_1(\mathbf{x}, t)$ and $g_2(\mathbf{x})$ are given sufficiently smooth functions, $\partial\Omega = \Gamma_1 \bigcup \Gamma_2$ is the closed curve bounding the region Ω. The boundary condition (4) is variable separable, with spatial dependence given by $g_2(\mathbf{x})$ and time dependence given by $h(t)$. Here ${}^cD_t^\alpha$ $(0 < \alpha < 1)$ denotes the Caputo fractional derivative of order α with respect to t and it is defined by [15]

$${}^cD_t^\alpha u(\mathbf{x}, t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\partial u(\mathbf{x}, \eta)}{\partial \eta} \frac{d\eta}{(t-\eta)^\alpha}, \quad 0 < \alpha < 1,$$

where $\Gamma(\cdot)$ is the Gamma function.

The condition (5) is encountered in many important applications in heat transfer, thermoelasticity, and industry, see [16–19]. Because of the importance of this type of equations in science and engineering, in the recent decades there had been growing interest in development of computational techniques for the numerical solution of non-classical boundary values problems. Specially, the numerical investigation of the two-dimensional diffusion equation with non-classical boundary conditions have been considered by a number of authors, see, e.g., [20–25]. In papers [23,24], Dehghan developed the implicit and explicit finite-difference schemes for the two-dimensional parabolic problems with non-classical boundary conditions. The alternating direction implicit (ADI) schemes have been investigated in [22,25]. In [26], Sajavičius constructed a weighted splitting finite difference approximation of a two-dimensional parabolic equation with non-local integral conditions. However, the application of the finite difference scheme to irregular domains seems to be not straightforward. Recently, Abbasbandy et al. [20,21] developed the meshless local Petrov–Galerkin method (MLPG) based on the moving least squares to obtain approximate solution of the non-classical diffusion equation with Dirichlet and Neumann boundary conditions. Kazem and Rad [27] have presented and applied a meshless method based on the radial basis functions (RBFs) for the non-local boundary value problem with Neumann boundary conditions. A comparison between meshless local weak and strong forms based on particular solutions for a non-classical diffusion model has also been undertaken by Abbasbandy et al. [28]. Their results are quite encouraging. However, all above mentioned papers dealt with the integer order differential equations. As we have known, the work done on the numerical solution of the TFDE with non-local boundary conditions is relatively sparse.

In the present paper, we investigate a numerical scheme based on the method of approximate particular solutions (MAPS) [29] using multiquadratic radial basis function (MQ-RBF) [30,31] for solving time-fractional diffusion equation with non-local boundary conditions. Results for several numerical examples are presented to demonstrate the efficacy of the new scheme. It should be noted that the MAPS using MQ-RBF contains a user defined shape parameter, c, which affects the stability and accuracy of the solution. The accuracy of the solution continues to improve as c increases; however, when c is large, the accuracy gets worse and eventually breaks down [32,33]. Thus, the value of the shape parameter has to be selected carefully. Almost existed approach work in time independent problems with a fixed c. Hence, the challenge of how to choose optimal shape parameter remains untouched. In this paper, we apply a leave-one-out cross validation (LOOCV) algorithm proposed by Rippa [34] to select a good value of shape parameter c for MAPS.

The outline of the paper is as follows. In Section 2, we briefly describe the MAPS which is used for solving the time-fractional diffusion equations, and how to compute the good value of the shape parameter. In Section 3, we present the results for several numerical examples to demonstrate the stability and high accuracy of the numerical algorithm. Finally, concluding remarks are given in Section 4.

2. Methodology

In this section, we describe the numerical scheme for solving the time-fractional diffusion equation with a non-local boundary condition.

2.1. Time fractional derivative discretization

To illustrate how to apply the MAPS as a spatial meshless scheme to solve the problem (1)–(5), we first reduce the above time-fractional diffusion equation into a series of elliptic PDEs using the finite difference approximation to discretize the time-fractional derivative. We note here that there are other reduction techniques such as the Laplace transform and the Fourier transform that can achieve the same purpose.

Suppose the time interval $[0, T]$ is discretized uniformly into K subintervals; define $t_k = k\Delta t$, $k = 0, 1, \ldots, K$, where $\Delta t = T/K$ is the time step. Let $u(\mathbf{x}, t_k)$ be the exact value of a function $u(\mathbf{x}, t)$ at time step t_k. Then, the time fractional derivative at $t = t_{k+1}$ can be approximated [8]

$${}^cD_t^\alpha u(\mathbf{x}, t_{k+1}) = \frac{(\Delta t)^{-\alpha}}{\Gamma(2-\alpha)} \sum_{j=0}^{k} \omega_j [u(\mathbf{x}, t_{k-j+1}) - u(\mathbf{x}, t_{k-j})] + \mathcal{O}((\Delta t)^{2-\alpha}), \tag{6}$$

for $k = 0, 1, \ldots, K-1$ where the weight is defined as $\omega_j = (j+1)^{1-\alpha} - j^{1-\alpha}$, $j = 0, 1, \ldots, k$.