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Error estimates for spectral approximation of elliptic control problems with integral state and control constraints[☆]

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ABSTRACT

The goal of this paper is to investigate the Legendre–Galerkin spectral approximation of elliptic optimal control problems with integral state and control constraints. Thanks to the appropriate base functions of the discrete spaces, the discrete system is with sparse coefficient matrices. We first present the optimality conditions of the control system. Then a priori and a posteriori error estimates both in H^1 and L^2 norms are derived. Some numerical tests indicate that the spectral accuracy can be achieved, and the proposed method is competitive for solving control problems.

1. Introduction

In this paper, we discuss the following optimal control problem:

$$\begin{cases} \min J(u, y) = \dfrac{1}{2}\|y(u) - y_0\|_{0,\Omega}^2 + \dfrac{\alpha}{2}\|u\|_{0,\Omega}^2, \\ \text{subject to } -\Delta y(u) = u \quad \text{in } \Omega, \qquad y(u) = 0 \quad \text{on } \partial\Omega, \\ (u, y) \in U_{ad} \times K, \end{cases} \tag{1.1}$$

where U_{ad} and K are closed convex sets, which in fact represents lots of practical and useful optimal control problems. It becomes a purely control-constrained problem when K equals the whole state space. One can find a lot of work on the numerical methods for this class of control problems. With the successful application in solving PDE numerically, the finite element method has been widely used to compute control-constrained problems, see, for example, [1–5]. It reduces to a purely state-constrained problem when U_{ad} equals the whole control space, which is frequently met in many applications. Generally, control problems with state constraints are more difficult to deal with than ones with control constraints. Extensive research has been carried out on various theoretical aspects and numerical strategies. Casas provides the techniques for deriving optimality conditions and regularity results for state-constrained elliptic control problems in [6,7]. We refer to [8,9] for more results. On the numerical methods, Bergounioux and Kunisch take an augmented Lagrangian method to solve state and control constrained problems in [10], and primal–dual active set algorithm to compute

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state constrained problems in [11]. Semi-smooth Newton methods are proposed to approximate state-constrained control problems in [12,13]. The state-constrained optimal control is approximated by a sequence of control-constrained problems reformulated from the original problem with state constraint in [14]. [15] proposes a mixed variational scheme for control problems with pointwise state constraints, and a direct numerical algorithm is adopted without the optimality conditions. [16] derives a posteriori error analysis of finite element approximation for distributed elliptic control problems with pointwise state constraints. In [17,18], the authors consider the finite element approximation of integral state constrained optimal control problems. In [19,20], the authors derive a priori error estimates and equivalent residual-type a posteriori error estimators of finite element approximation for flow control problems with L^2 state constraint.

In recent years, the spectral method has been used to approximate optimal control problems. With the global polynomials as the trial functions, the spectral method can provide very accurate approximation and has been successfully applied in numerical solutions of PDE, especially in the field of computational fluid dynamics (see, for example, [21–24], and the references therein). Generally, the solutions to the optimal control problems have limited regularity due to, e.g., the constraints, which leads to lose of the spectral accuracy. Consequently the spectral method is not so widely applied in control problems in contrast with the finite element method. However, the spectral method enjoys a great superiority of fast convergence rate, high-order accuracy, and a relatively small number of unknowns when the solutions have the higher regularity, which is vital to efficient approximation of optimal control problems. There have been many researches on the spectral approximation of control problems. Both a priori and a posteriori error estimates are derived for the spectral approximation of elliptic control problems with integral control constraint in [25]. The numerical tests confirm the error estimates and indicate that the spectral accuracy is achieved, thanks to the higher regularity of the optimal control. In [26], the flow optimal control with integral control constraint is successfully approximated by the Legendre–Galerkin spectral method, where both the unconstrained and constrained cases are discussed. In [27], the spectral method is used to approximate state constrained control problems governed by the first bi-harmonic equation. These show that the spectral method has gained increasing popularity in approximating control problems governed by PDEs in the last decade.

As to the control problems with state and control constraints, there have been some studies on both theoretical analysis and numerical approximation in the literature. In [6], Casas proves the existence of Lagrange multiplier and provides the optimality conditions for the control problems with state and control constraints, then gives some regularity results. Rösch and Tröltzsch investigate a semilinear parabolic control problem with pointwise control-state constraints, where the Lagrange multipliers can be assumed to be bounded and measurable functions and a second-order sufficient optimality condition is derived that considers strongly active constraints in [28]. A semilinear elliptic control problem with mixed control-state constraints is investigated in [29], where the existence of bounded and measurable Lagrange multipliers is proven. In [30], Casas investigates the finite element approximation of control problems governed by semilinear elliptic equations with bound constraints on the control and finitely many state constraints. In [31], the finite element method is used to approximate an elliptic control problem with state and control constraints, and a posteriori error estimator that contains only computable quantities is provided. We can find more theoretical and numerical results in [32–34]. However most of these researches concentrate on control problems with pointwise constraints. In many applications, we take more attention to the average value or some energy-norm of the state or control variable, and make well development on it. To our best knowledge there has been a lack of discussion on the spectral approximation of control problems with integral state and control constraints, which is the goal of this paper.

Let us shortly describe the structure of our paper. Spectral approximation and optimality conditions are presented in Section 2. A priori error estimates for the control problem are derived in Section 3. A posteriori error estimates are investigated in Section 4. Finally, two numerical experiments are presented to illustrate the effectiveness of the spectral approximation in Section 5.

Let $\Omega \subset \mathbb{R}^s$ $(s = 1, 2)$ be an interval or a rectangular domain. In this paper we adopt the standard notation $W^{m,q}(\Omega)$ for Sobolev spaces on Ω with the norm $\|\cdot\|_{W^{m,q}(\Omega)}$ and the seminorm $|\cdot|_{W^{m,q}(\Omega)}$ (or $\|\cdot\|_{m,q,\Omega}$, $|\cdot|_{m,q,\Omega}$ for simplification). We set $W_0^{m,q}(\Omega) \equiv \{w \in W^{m,q}(\Omega) : \ w|_{\partial\Omega} = 0\}$. We denote $W^{m,2}(\Omega)$ ($W_0^{m,2}(\Omega)$) by $H^m(\Omega)$ ($H_0^m(\Omega)$). In addition, C denotes a general positive constant independent of N, the order of the spectral approximation.

2. Spectral approximation and optimality conditions

The optimal control problem, its spectral approximation, and the optimality conditions are stated in this section. Let $Y = H_0^1(\Omega)$, $U = L^2(\Omega)$, we will investigate the following elliptic control problem with state and control constraints: find $(u, y) \in U \times Y$ such that

$$\begin{cases} \min J(u, y) = \dfrac{1}{2}\|y(u) - y_0\|_{0,\Omega}^2 + \dfrac{\alpha}{2}\|u\|_{0,\Omega}^2, \\ \text{subject to } -\Delta y(u) = u \quad \text{in } \Omega, \qquad y(u) = 0 \quad \text{on } \partial\Omega, \\ (u, y) \in U_{ad} \times K, \end{cases} \tag{2.1}$$

where

$$U_{ad} = \left\{ u \in U : \int_\Omega u \geq 0 \right\}, \qquad K = \left\{ w \in L^1(\Omega) : \int_\Omega w \geq 0 \right\},$$