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Mixed finite elements for electromagnetic analysis

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ABSTRACT

The occurrence of spurious solutions is a well-known limitation of the standard nodal finite element method when applied to electromagnetic problems. The two commonly used remedies that are used to address this problem are (i) The addition of a penalty term with the penalty factor based on the local dielectric constant, and which reduces to a Helmholtz form on homogeneous domains (regularized formulation); (ii) A formulation based on a vector and a scalar potential. Both these strategies have some shortcomings. The penalty method does not completely get rid of the spurious modes, and both methods are incapable of predicting singular eigenvalues in non-convex domains. Some non-zero spurious eigenvalues are also predicted by these methods on non-convex domains. In this work, we develop mixed finite element formulations which predict the eigenfrequencies (including their multiplicities) accurately, even for nonconvex domains. The main feature of the proposed mixed finite element formulation is that no ad-hoc terms are added to the formulation as in the penalty formulation, and the improvement is achieved purely by an appropriate choice of finite element spaces for the different variables. We show that the formulation works even for inhomogeneous domains where 'double noding' is used to enforce the appropriate continuity requirements at an interface. For two-dimensional problems, the shape of the domain can be arbitrary, while for the three-dimensional ones, with our current formulation, only regular domains (which can be nonconvex) can be modeled. Since eigenfrequencies are modeled accurately, these elements also yield accurate results for driven problems.

1. Introduction

One of the major problems in computational electromagnetics using conventional nodal finite elements has been the occurrence of spurious solutions. In structural mechanics, mesh refinement helps to reduce the errors; however, in the electromagnetics setting, it merely increases the number of spurious modes. Edge elements [1–5], which use basis function associated with each edge are widely used to circumvent this difficulty. These elements ensure tangential continuity of the field along an element edge and model the null space of curl operator accurately. They can model both, singularities and inhomogeneous domains. However, they also have their limitations [6,7]. Since the normal component is discontinuous across element faces even for homogeneous domains, the efficiency is reduced. Another disadvantage is that coupling with structural or thermal variables (where nodal finite elements are used) in multiphysics problems could be difficult.

In order to deal with spurious modes within the framework of the nodal finite element method, a penalty function or regularization method [8–14] is used, where a term involving the divergence of the electric field is added to the original variational formulation. However, this method does not eliminate spurious modes, and in fact just pushes them

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towards the higher part of the spectrum; the problems with this method are particularly severe when non-convex and/or inhomogeneous domains are involved. Costabel and Dauge [15] and Otin [11] have proposed a weighted penalty method where a weight is appended to the penalty parameter which tends to zero at a field singularity. An alternative approach is the use of potentials [16–18] along with the choice of a suitable gauge. Even in this method, a penalty-type term has to be added. The method is quite robust even for inhomogeneous domains. However, because of the presence of the penalty term, it cannot predict the singular eigenvalues on nonconvex domains.

Recent works include using a combination of cubic Hermite splines and quadratic Lagrange interpolations [19], and a least squares finite element method [20]. The former method can be applied only to regular geometries, while the latter uses nonstandard elements with stabilizing face bubble functions.

In this work, we develop nodal-based mixed finite element formulations for two and three-dimensional problems. The two-dimensional elements yield very accurate approximations of the eigenvalues, including the correct multiplicities, and including singular eigenvalues for non-convex domains. Non-homogeneous and curved domains can also be modeled with these elements. The three-dimensional elements can currently be applied only to Cartesian geometries (including domains with singularities and inhomogeneous domains); further work is required to extend their capabilities to non-regular geometries. The main feature of the proposed mixed formulation is that *no additional* terms, such as the penalty terms that are used in regularized or potential-based formulations, are added to the variational formulation, and in addition, there are no parameters that have to be chosen by the user; alternative formulations which a-priori eliminate spurious modes are presented in Refs. [13,21]. The improvement is achieved simply by using appropriately chosen interpolations for the various fields. The other main feature is that standard C^0 Lagrange interpolations are used for all the fields, and standard Gaussian quadrature is used to compute all the matrices.

The outline of the remainder of the article is as follows. We first briefly review the variational and finite element formulations for the potential method. Next we discuss the variational formulation and interpolations for the proposed mixed finite elements. Finally, we show the high accuracy and robustness of the proposed formulation by comparing the solutions obtained against either analytical or benchmark solutions obtained using existing numerical strategies.

2. Mathematical formulation

2.1. Maxwell equations in electromagnetics

The strong form of the Maxwell equations is [22]

$$\frac{\partial \boldsymbol{B}}{\partial t} + \nabla \times \boldsymbol{E} = \boldsymbol{0}, \tag{1a}$$

$$\nabla \cdot \boldsymbol{B} = 0, \tag{1b}$$

$$\frac{\partial \boldsymbol{D}}{\partial t} - \nabla \times \boldsymbol{H} = -\boldsymbol{j}, \tag{1c}$$

$$\nabla \cdot \boldsymbol{D} = \rho, \tag{1d}$$

where $\boldsymbol{E}$ and $\boldsymbol{H}$ are the electric and magnetic fields, $\boldsymbol{D}$ is the electric displacement (electric flux), $\boldsymbol{B}$ is the magnetic induction (magnetic flux), ρ is the charge density and $\boldsymbol{j}$ is the current density. The above governing equations are supplemented by the constitutive relations

$$\boldsymbol{D} = \epsilon \boldsymbol{E}, \tag{2a}$$

$$\boldsymbol{B} = \mu \boldsymbol{H}, \tag{2b}$$

where ϵ and μ are the electric permittivity and magnetic permeability, respectively. Substituting the constitutive relations into Eqs. (1a), (1c) and (1d), and assuming that ϵ and μ are independent of time, we get

$$\frac{\partial \boldsymbol{H}}{\partial t} + \frac{1}{\mu} \nabla \times \boldsymbol{E} = \boldsymbol{0}, \tag{3}$$

$$\epsilon \frac{\partial \boldsymbol{E}}{\partial t} - \nabla \times \boldsymbol{H} = -\boldsymbol{j}, \tag{4}$$

$$\nabla \cdot (\epsilon \boldsymbol{E}) = \rho. \tag{5}$$

From Eqs. (1d), (2a) and (4), we get the compatibility condition

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \boldsymbol{j} = 0.$$