Available online at www.sciencedirect.com
ScienceDirect
Computers and Mathematics with Applications 55 (2008) 2498–2505
An International Journal computers & mathematics with applications
www.elsevier.com/locate/camwa

A general refinement of Jordan-type inequality

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Received 2 March 2006; received in revised form 20 September 2007; accepted 10 October 2007

Abstract

In this work, a general form of Jordan's inequality:

$$P_{2N}(x) + a_{N+1}(\pi^2 - 4x^2)^{N+1} \le \frac{\sin x}{x} \le P_{2N}(x) + \frac{1 - \sum_{n=0}^{N} a_n \pi^{2n}}{\pi^{2(N+1)}}(\pi^2 - 4x^2)^{N+1}$$

is established, where $x \in (0, \pi/2]$, $P_{2N}(x) = \sum_{n=0}^{N} a_n (\pi^2 - 4x^2)^n$, $a_0 = \frac{2}{\pi}$, $a_1 = \frac{1}{\pi^3}$, $a_{n+1} = \frac{2n+1}{2(n+1)\pi^2} a_n - \frac{1}{16n(n+1)\pi^2} a_{n-1}$, and $N \ge 0$ is a natural number. The applications of the above result give the general improvement of the Yang Le inequality and a new infinite series $(\sin x)/x = \sum_{n=0}^{\infty} a_n (\pi^2 - 4x^2)^n$ for $0 < |x| \le \pi/2$.

Keywords: Lower and upper bounds; A general form of Jordan's inequality; Yang Le inequality; The Spherical Bessel Functions (SBFs); The SBFs of the first kind $j_n(x) = \sqrt{\frac{\pi}{2x}} J_{n+\frac{1}{2}}(x)$; A new infinite series for $(\sin x)/x$

1. Introduction

The following result is known as Jordan's inequality [1]:

Theorem 1. *If $0 < x \le \pi/2$, then*

$$\frac{2}{\pi} \le \frac{\sin x}{x} < 1 \tag{1}$$

with equality if and only if $x = \pi/2$.

Debnath and Zhao [2] have obtained a new lower bound for the function $\frac{\sin x}{x}$. Their result reads as follows

Theorem 2. *If $0 < x \le \pi/2$, then*

$$\frac{\sin x}{x} \ge \frac{2}{\pi} + \frac{1}{\pi^3}(\pi^2 - 4x^2) \tag{2}$$

with the equality if and only if $x = \pi/2$.

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doi:10.1016/j.camwa.2007.10.004

The author of this paper [3] obtained a further result:

Theorem 3. *If* $0 < x \leq \pi/2$, *then*

$$\frac{2}{\pi} + \frac{1}{\pi^3}(\pi^2 - 4x^2) \leq \frac{\sin x}{x} \leq \frac{2}{\pi} + \frac{\pi - 2}{\pi^3}(\pi^2 - 4x^2) \tag{3}$$

with the equalities if and only if $x = \pi/2$. *Furthermore,* $\frac{1}{\pi^3}$ *and* $\frac{\pi-2}{\pi^3}$ *are the best constants in* (3).

Recently, the author of this paper [4] has given new improvement of Jordan's inequality.

Theorem 4. *If* $0 < x \leq \pi/2$, *then*

$$\frac{2}{\pi} + \frac{1}{\pi^3}(\pi^2 - 4x^2) + \frac{12 - \pi^2}{16\pi^5}(\pi^2 - 4x^2)^2 \leq \frac{\sin x}{x} \leq \frac{2}{\pi} + \frac{1}{\pi^3}(\pi^2 - 4x^2) + \frac{\pi - 3}{\pi^5}(\pi^2 - 4x^2)^2 \tag{4}$$

with the equalities if and only if $x = \pi/2$. *Furthermore,* $\frac{12-\pi^2}{16\pi^5}$ *and* $\frac{\pi-3}{\pi^5}$ *are the best constants in* (4).

In the form of (3) and (4), we finally obtain the general Jordan's inequality as follows

Theorem 5. *Let* $0 < x \leq \pi/2$ *and* $N \geq 0$ *is a natural number, then*

$$P_{2N}(x) + a_{N+1}(\pi^2 - 4x^2)^{N+1} \leq \frac{\sin x}{x} \leq P_{2N}(x) + \frac{1 - \sum_{n=0}^{N} a_n \pi^{2n}}{\pi^{2(N+1)}}(\pi^2 - 4x^2)^{N+1} \tag{5}$$

with the equalities if and only if $x = \pi/2$, *where,* $P_{2N}(x) = \sum_{n=0}^{N} a_n(\pi^2 - 4x^2)^n$ *and*

$$a_0 = \frac{2}{\pi}, \qquad a_1 = \frac{1}{\pi^3}, \qquad a_{n+1} = \frac{2n+1}{2(n+1)\pi^2}a_n - \frac{1}{16n(n+1)\pi^2}a_{n-1}, \quad n = 1, 2, \ldots. \tag{6}$$

Furthermore, a_{N+1} *and* $\frac{1-\sum_{n=0}^{N} a_n\pi^{2n}}{\pi^{2(N+1)}}$ *are the best constants in* (5).

2. Six lemmas

Lemma 1 (*[5,6]*). *Let* $f, g : [a, b] \to \mathbb{R}$ *be two continuous functions which are differentiable on* (a, b). *Further, let* $g' \neq 0$ *on* (a, b). *If* f'/g' *is increasing (or decreasing) on* (a, b), *then the functions*

$$\frac{f(x) - f(b)}{g(x) - g(b)}$$

and

$$\frac{f(x) - f(a)}{g(x) - g(a)}$$

are also increasing (or decreasing) on (a, b).

Lemma 2 (*[7,8]*). *Let* $j_n(x)$ *be the Spherical Bessel Functions of the first kind,* $j_n(x) = \sqrt{\frac{\pi}{2x}} J_{n+\frac{1}{2}}(x)$, *then*

$$j_n(x) = (-x)^n \left[\frac{1}{x}\frac{\mathrm{d}}{\mathrm{d}x}\right]^n \frac{\sin x}{x}, \quad n = 0, 1, 2, \ldots. \tag{7}$$

Lemma 3 (*[9,10]*). *Let* $j_n(x)$ *be the SBFs of the first kind,* $j_n(x) = \sqrt{\frac{\pi}{2x}} J_{n+\frac{1}{2}}(x)$, *then*

$$j_{n+1}(x) = \frac{2n+1}{x} j_n(x) - j_{n-1}(x) \tag{8}$$

or

$$(2n+1) j_n(x) = x[j_{n+1}(x) + j_{n-1}(x)]. \tag{9}$$