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Existence and uniqueness of common fixed points for two multivalued operators in ordered metric spaces

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ABSTRACT

In this paper, we discuss some new fixed point theorems for a pair of multivalued operators which satisfy weakly generalized contractive conditions. Our results are the extension and improvement of corresponding results of [J. Harjani and K. Sadarangani, Fixed point theorems for weakly contractive mappings in partially ordered sets, Nonlinear Analysis, 71 (2009) 3403–3410] and [X. Zhang, Common fixed point theorems for some new generalized contractive type mappings, J. Math. Anal. Appl. 333 (2007) 780–786]. Finally, some examples are given to illustrate the usability of our results.

1. Introduction

It is well known that the contractive-type conditions are very important in the study of fixed point theory. The first important result on fixed points for contractive-type mappings was the well-known Banach–Caccioppoli theorem, published for the first time in 1922 in [1] and also found in [2]. Then Kannan analyzed a substantially new type of contractive condition in [3]. Nadler in [4] extended the contraction into multivalued mappings and obtained the existence of fixed points. Since then there have been many theorems dealing with mappings satisfying various types of contractive inequality, we refer to [5–19] and references therein. Very recently results of common fixed points for a pair of single-valued operators were obtained by applying various types of contractive conditions, we refer to [20–24]. Moreover, in [25–28] authors considered the analogy of multivalued mappings. For example, in [26], the existence of common fixed points for multivalued mappings was also considered recently by applying the monotone method in ordered Banach spaces. However, as far as our knowledge, few corresponding results of common fixed points for multivalued operators satisfying generalized contractive conditions are concerned (see [27,28]). The purpose of the present paper is to establish the common fixed point theorems for weakly contractive multivalued operators in ordered complete metric spaces. The weakly contractive single-valued maps were first defined by Alber and Guerre–Delabriere in [29]. Here we give a brief description of the basic known notions.

Let $(E, \|\cdot\|)$ be a Banach space, a selfmap F of E is said to satisfy the Banach contraction principle if there exists a constant k with $0 \le k < 1$ such that, for $x, y \in E$,

$$\|Fx - Fy\| \le k\|x - y\|.$$

As noted in the introduction of [29], this inequality can be written in the form

$$\|Fx - Fy\| \le \|x - y\| - q\|x - y\|,$$

where $k = 1 - q$ with $q \in (0, 1]$. The extension of the above inequality in the context of Banach spaces to what we called weakly contractive maps is a natural one. A selfmap F of E is said to be weakly contractive if

$$\|Fx - Fy\| \le \|x - y\| - \psi(\|x - y\|)$$

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for every $x, y \in E$, where $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ with $\mathbb{R}_+ = [0, \infty)$ is a continuous and nondecreasing function such that it is positive in $(0, \infty)$, $\psi(0) = 0$ and $\lim_{t\to\infty} \psi(t) = \infty$ (it is clear that φ needs to satisfy $\varphi(t) \le t$ for $t > 0$). [30] extended the notion to a metric space E, that is, a map $F : E \to E$ is said to be weakly contractive if

$$d(Fx, Fy) \le d(x, y) - \psi(d(x, y))$$

for all $x, y \in E$, where $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ satisfies the above mentioned conditions.

Let $(E, d, \le)$ denote an ordered complete metric space with a partial order $\le$ and distance $d(\cdot, \cdot)$. Let $d = \sup\{d(x, y) : x, y \in E\}$. Set $a = d$ if $d = \infty$ and $a > d$ if $d < \infty$. Moreover, [16] extended the notion in [30] to the weaker contraction for the multivalued operators, namely, the multivalued mapping $G : E \to 2^E$, $f \in \mathscr{F}[0, a)$ and $\varphi \in \Phi[0, f(a - 0))$ satisfy

$$f(H_d(Gx, Gy)) \le f(d(x, y)) - \varphi(f(d(x, y)))$$

for all $x, y \in E$ with x and y comparable. Then G is called a weakly generalized contraction with respect to f and φ (for the notations appear here we refers to the below definitions).

In this paper we will define an analogical weakly contractive type condition for two multivalued maps. Moreover, we will obtain some results which are also new even to the single-valued case of operator equations.

The rest of the paper is organized as follows. Section 2 deals with the preliminaries needed in the sequel. Section 3 establishes the main fixed point theorems and some corollaries in the applicable form to differential and integral inclusions and equations. To show the applicability of our results, in Section 4 we discusses several examples.

2. Preliminaries

In this paper, unless otherwise mentioned, let $(E, d, \le)$ denote an ordered complete metric space with a partial order $\le$ and distance $d(\cdot, \cdot)$. Let 2^E denote the family consisting of all nonempty subsets of E. The following hypothesis in E will be applied:

(H1) If $\{x_n\}$ is a nondecreasing (resp. nonincreasing) sequence in E such that $x_n \to x$, then $x_n \le x$ (resp. $x_n \ge x$) for all $n \in \mathbb{N}$.

We define the Hausdorff pseudometric in 2^E by $H_d : 2^E \times 2^E \to \mathbb{R}_+ \cup \{\infty\}$ given by

$$H_d(C, D) = \max\left\{\sup_{a\in C} d(a, D), \sup_{b\in D} d(C, b)\right\},$$

where $d(C, b) = \inf_{a\in C} d(a, b)$, $d(a, D) = \inf_{b\in D} d(a, b)$.

Definition 2.1 (*[16]*)**.** Let E be a metric space. A subset $D \subset E$ is said to be approximative if the multivalued mapping

$$\mathscr{P}_D(x) = \{y \in D : d(x, y) = d(D, x)\}, \quad \forall x \in E$$

has nonempty values.

The multivalued mapping $G : E \to 2^E$ is said to have approximative values, AV for short, if Gx is approximative for each $x \in E$.

The multivalued mapping $G : E \to 2^2$ is said to have comparable approximative values, CAV for short, if G has approximative values and, for each $z \in E$, there exists $y \in \mathscr{P}_{Gz}(x)$ such that y is comparable to z.

The multivalued mapping $G : E \to 2^E$ is said to have upper comparable approximative values, UCAV, for short (resp. lower comparable approximative values, LCAV for short) if G has approximative values and, for each $z \in E$, there exists $y \in \mathscr{P}_{Gz}(x)$ such that $y \ge z$ (resp. $y \le z$).

It is clear that G has approximative values if it has compact values. In addition, if G is single-valued, then UCAV (LCAV) means that $Gx \ge x$ ($Gx \le x$) for $x \in E$.

Definition 2.2. The multivalued mapping G is said to have a fixed point if there is $x \in E$ such that $x \in Gx$.

In what follows, we give an analogy of the contraction which is called the weakly generalized contractive type condition for multivalued mappings which will play an important role in this sequel. To this end, we first introduce the following functions.

Let $a \in (0, \infty]$, $R_a^+ = [0, a)$. Let $f : R_a^+ \to \mathbb{R}$ satisfy

(i) $f(0) = 0$ and $f(t) > 0$ for each $t \in (0, a)$;
(ii) f is nondecreasing on R_a^+;
(iii) f is continuous.
(iv) $f(t + s) \le f(t) + f(s)$ for $s, t \in R_a^+$.

For examples of such function f we refer to [15]. Define $\mathscr{F}[0, a) = \{f | f$ satisfies (i)–(iv) above$\}$. It is easy to see that $\lim_{n\to\infty} f(t_n) = 0$ for $t_n \in R_a^+$, then $\lim_{n\to\infty} t_n = 0$ if $f \in \mathscr{F}[0, a)$.

Let $a \in (0, \infty]$, $\varphi : R_a^+ \to \mathbb{R}_+$ satisfy

(i) $\varphi(0) = 0$ and $\varphi(t) > 0$ for each $t \in (0, a)$.