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# Integral transforms of functions to be in certain class defined by the combination of starlike and convex functions

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ABSTRACT

Let $P_\gamma(\beta)$, $\beta < 1$, denote the class of all normalized analytic functions $f$ in the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ such that

$$\mathrm{Re}\left(e^{i\phi}\left((1-\gamma)\frac{f(z)}{z} + \gamma f'(z) - \beta\right)\right) > 0, \quad z \in \mathbb{D}$$

for some $\phi \in \mathbb{R}$. Let $M(\mu, \alpha)$, $0 \le \mu < 1$, denote the Pascu class of $\alpha$-convex functions of order $\mu$ and given by the analytic condition

$$\mathrm{Re}\,\frac{\alpha z(zf'(z))' + (1-\alpha)zf'(z)}{\alpha zf'(z) + (1-\alpha)f(z)} > \mu$$

which unifies $S^*(\mu)$ and $C(\mu)$, respectively, the classes of analytic functions that map $\mathbb{D}$ onto the starlike and convex domain. In this work, we consider integral transforms of the form

$$V_\lambda(f)(z) = \int_0^1 \lambda(t)\frac{f(tz)}{t}\,dt.$$

The aim of this paper is to find conditions on $\lambda(t)$ so that the above transformation carry $P_\gamma(\beta)$ into $M(\mu, \alpha)$. As applications, for specific values of $\lambda(t)$, it is found that several known integral operators carry $P_\gamma(\beta)$ into $M(\mu, \alpha)$.


## 1. Introduction and key lemmas

Let $\mathcal{A}$ denote the class of all functions $f$ analytic in the open unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ with the normalization $f(0) = f'(0) - 1 = 0$ and $\mathcal{S}$ be the class of functions $f \in \mathcal{A}$ that are univalent in $\mathbb{D}$. A function $f \in \mathcal{S}$ is said to be starlike or convex, if $f$ maps $\mathbb{D}$ conformally onto the domains, respectively, starlike with respect to origin and convex. Note that $f$ is convex in $\mathbb{D}$ if and only if $zf'$ is starlike in $\mathbb{D}$ follows from the well-known Alexander theorem (see [1] for details).

The generalization of these two classes are given by the following analytic characterizations;

$$S^*(\mu) := \left\{f \in \mathcal{A} : \mathrm{Re}\,\frac{zf'(z)}{f(z)} > \mu,\ 0 \le \mu < 1\right\}$$

$$K(\mu) := \left\{f \in \mathcal{A} : \mathrm{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \mu,\ 0 \le \mu < 1\right\},$$

so that $S^*(0) \equiv S^*$ and $K(\mu) \equiv K$ are the starlike and convex classes respectively.

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A function $f \in \mathcal{A}$ is said to be in the Pascu class of $\alpha$-convex functions of order $\mu$ $(0 \le \mu < 1)$ if [2]

$$\mathrm{Re}\frac{\alpha z(zf'(z))' + (1-\alpha)zf'(z)}{\alpha zf'(z) + (1-\alpha)f(z)} > \mu.$$

or in other words

$$\alpha zf'(z) + (1-\alpha)f(z) \in \mathcal{S}^*(\mu).$$

This class is denoted by $M(\alpha, \mu)$. Note that $M(0, \mu) = S^*(\mu)$ and $M(1, \mu) = K(\mu)$ which implies that $M(\alpha, \mu)$ is a smooth passage between the class of starlike and convex functions.

Further, $f \in \mathcal{A}$ is said to be close-to-convex in $\mathbb{D}$, with respect to a starlike function $g$, if $f$ satisfies the analytic characterization, $\mathrm{Re}\left(e^{i\sigma}\frac{zf'(z)}{g(z)}\right) > 0$, $z \in \mathbb{D}$, $\sigma \in \mathbb{R}$. These close-to-convex functions $f$ satisfy a nice geometric property that the complement of image of $\mathbb{D}$ under $f$ are the union of closed halflines such that the corresponding open halflines are disjoint [3, Theorem 2.12, p. 52]. We denote by $\mathcal{C}$ the class of all close-to-convex functions in $\mathbb{D}$.

The main objective of this work is to find conditions on the non-negative real valued integrable function $\lambda(t)$ satisfying $\int_0^1 \lambda(t)dt = 1$, such that the operator

$$F(z) = V_\lambda(f)(z) := \int_0^1 \lambda(t)\frac{f(tz)}{t}dt \tag{1.1}$$

is in the class $M(\alpha, \mu)$. Note that this operator was introduced in [4]. To investigate this admissibility property the class to which the function $f$ belongs is important. Let $P_\gamma(\beta)$, $\beta < 1$, denote the class of all normalized analytic functions $f$ in the unit disc $\mathbb{D}$ such that

$$\mathrm{Re}\left(e^{i\phi}\left((1-\gamma)\frac{f(z)}{z} + \gamma f'(z) - \beta\right)\right) > 0, \quad z \in \mathbb{D}$$

for some $\phi \in \mathbb{R}$. This class and its particular cases were considered by many authors to prove that the operator given by (1.1) is univalent under certain conditions and in $M(\alpha, \mu)$ for some particular values of $\alpha$ and $\mu$. This work was motivated in [4] by studying the conditions under which $V_\lambda(P_1(\beta)) \subset M(0, 0)$ and generalized in [5] by studying the case $V_\lambda(P_\gamma(\beta)) \subset M(0, 0)$. In [6] the conditions under which $V_\lambda(P_1(\beta)) \subset M(1, 0)$ were studied. An extensive study of $V_\lambda(P_\gamma(\beta))$ to the class $M(0, \mu)$ is in [7] and to the class $M(1, \mu)$ is in [8].

One of the main tools in the objective of this work is the following. If $f$ and $g$ are in $\mathcal{A}$ with the power series expansions $f(z) = \sum_{k=0}^{\infty} a_k z^k$ and $g(z) = \sum_{k=0}^{\infty} b_k z^k$ respectively, then the convolution or Hadamard product of $f$ and $g$ is given by $h(z) = \sum_{k=0}^{\infty} a_k b_k z^k$.

For $\Lambda : [0, 1] \to \mathbb{R}$ integrable over $[0, 1]$ and positive on $(0, 1)$, let

$$L_\Lambda(f) := \inf_{z\in\Delta} \int_0^1 \Lambda(t)\left(\mathrm{Re}\frac{f(tz)}{tz} - \frac{1}{(1+t)^2}\right)dt, \quad f \in \mathcal{C}, \tag{1.2}$$

and

$$L_\Lambda(\mathcal{C}) = \inf_{f\in\mathcal{C}} L_\Lambda(f). \tag{1.3}$$

Fournier and Ruscheweyh [4] have established the following:

**Theorem 1.1.** (i) *If* $\frac{\Lambda(t)}{1-t^2}$ *is decreasing on* $(0, 1)$ *then* $L_\Lambda(\mathcal{C}) = 0$.

(ii) *If* $\lambda : [0, 1] \to \mathrm{Re}\, R$ *is non-negative with* $\int_0^1 \lambda(t)dt = 1$, $\Lambda(t) = \int_t^1 \lambda(t)\frac{dt}{t}$ *satisfies* $t\Lambda(t) \to 0$ *for* $t \to 0+$ *and*

$$\mathcal{V}_\lambda(f) = \int_0^1 \lambda(t)\frac{f(tz)}{t}dt, \quad f \in \mathcal{A}, \tag{1.4}$$

*then for* $\beta_\lambda < 1$ *given by*

$$\frac{\beta_\lambda}{1-\beta\lambda} = -\int_0^1 \lambda(t)\frac{1-t}{1+t}dt, \tag{1.5}$$

*we have for* $\beta = \beta_\lambda$ : $\mathcal{V}_\lambda(\mathcal{P}_\beta) \subset \mathcal{S}$ *and*

$$\mathcal{V}_\lambda(\mathcal{P}_\beta) \subset \mathcal{S}^* \Leftrightarrow L_\Lambda\mathcal{C} = 0.$$

*For* $\beta < \beta_\lambda$ *there exists* $f \in \mathcal{P}_\beta$ *with* $\mathcal{V}_\lambda(f)$ *not even locally univalent.*