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An extension of coupled fixed point's concept in higher dimension and applications

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A B S T R A C T

We introduce the concept of fixed point of N-order for mappings $F : X^N \to X$, where $N \geq 2$ and X is an ordered set endowed with a metric d. We establish fixed point results for such mappings satisfying a given contractive condition. Presented theorems extend and generalize the coupled fixed point results of Bhaskar and Lakshmikantham [T. Gnana Bhaskar, V. Lakshmikantham, Fixed point theorems in partially ordered metric spaces and applications, Nonlinear Anal. 65 (7) (2006) 1379–1393] and the tripled fixed point results of Berinde and Borcut [V. Berinde, M. Borcut, Tripled fixed point theorems for contractive type mappings in partially ordered metric spaces, Nonlinear Anal. 74 (2011) 4889–4897]. Some applications to integral equations and to matrix equations are also presented in this work.

1. Introduction

In recent years there has been a growing interest in studying the existence of fixed points for contractive mappings satisfying monotone properties in ordered metric spaces. This trend was initiated by Ran and Reurings in [1] where they extended the Banach contraction principle in partially ordered sets with some applications to matrix equations. Ran and Reurings' fixed point theorem was further extended and refined by many authors, e.g. [2–11].

In [12], Bhaskar and Lakshmikantham introduced the concept of coupled fixed point for contractive operators $F : X \times X \to X$ satisfying the mixed monotone property, where X is a partially ordered metric space, and then established some interesting coupled fixed point theorems. They also illustrated these important results by proving the existence and uniqueness of the solution for a periodic boundary value problem.

Definition 1.1 (*Bhaskar and Lakshmikantham [12]*)**.** Let $(X, \preceq)$ be a partially ordered and $F : X \times X \to X$. We say that F has the mixed monotone property if $F(x_1, x_2)$ is monotone nondecreasing in x_1 and is monotone non increasing in x_2.

Definition 1.2 (*Bhaskar and Lakshmikantham [12]*)**.** An element $(x_1, x_2) \in X \times X$ is called a coupled fixed point of $F : X \times X \to X$ if

$$x_1 = F(x_1, x_2) \quad \text{and} \quad x_2 = F(x_2, x_1).$$

We can now state the main results in [12].

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Theorem 1.1 (*Bhaskar and Lakshmikantham [12]*). *Let $(X, \preceq)$ be a partially ordered set and suppose there is a metric d on X such that (X, d) is a complete metric space. Let $F : X \times X \to X$ be a continuous mapping having the mixed monotone property. Assume that there exists a $k \in [0, 1)$ such that*

$$d(F(x_1, x_2), F(y_1, y_2)) \leq \frac{k}{2}[d(x_1, y_1) + d(x_2, y_2)] \tag{1.1}$$

for all $x_1 \preceq y_1$ and $x_2 \succeq y_2$. If there exists $x_1^{(0)}, x_2^{(0)} \in X$ such that $x_1^{(0)} \preceq F(x_1^{(0)}, x_2^{(0)})$ and $x_2^{(0)} \succeq F(x_2^{(0)}, x_1^{(0)})$, then there exist $x_1, x_2 \in X$ such that

$$x_1 = F(x_1, x_2) \quad \text{and} \quad x_2 = F(x_2, x_1).$$

Theorem 1.2 (*Bhaskar and Lakshmikantham [12]*). *Let $(X, \preceq)$ be a partially ordered set and suppose there is a metric d on X such that (X, d) is a complete metric space. Assume that X has the following property:*

(i) *If a nondecreasing sequence $\{x^{(q)}\} \to x$, then $x^{(q)} \preceq x$ for all q,*
(ii) *If a non increasing sequence $\{y^{(q)}\} \to y$, then $y^{(q)} \succeq y$ for all q.*

Let $F : X \times X \to X$ be a mapping having the mixed monotone property. Assume that there exists a $k \in [0, 1)$ such that (1.1) is satisfied for all $x_1 \preceq y_1$ and $x_2 \succeq y_2$. If there exists $x_1^{(0)}, x_2^{(0)} \in X$ such that $x_1^{(0)} \preceq F(x_1^{(0)}, x_2^{(0)})$ and $x_2^{(0)} \succeq F(x_2^{(0)}, x_1^{(0)})$, then there exist $x_1, x_2 \in X$ such that

$$x_1 = F(x_1, x_2) \quad \text{and} \quad x_2 = F(x_2, x_1).$$

Many generalizations and extensions of Theorems 1.1 and 1.2 exist in the literature. For more details, we refer the reader to [13–22]. In [14], Berinde and Borcut introduced the concept of tripled fixed point and established fixed point results for mappings having a monotone property and satisfying a contractive condition in ordered metric spaces. We summarize in the following the basic notions and results established in [14].

Definition 1.3 (*Berinde and Borcut [14]*). Let $(X, \preceq)$ be a partially ordered set and $F : X \times X \times X \to X$. We say that F has the mixed monotone property if $F(x_1, x_2, x_3)$ is monotone nondecreasing in x_1 and x_3, and is monotone non increasing in x_2.

Definition 1.4 (*Berinde and Borcut [14]*). An element $(x_1, x_2, x_3) \in X \times X \times X$ is called a tripled fixed point of $F : X \times X \times X \to X$ if

$$F(x_1, x_2, x_3) = x_1, \qquad F(x_2, x_1, x_2) = x_2 \quad \text{and} \quad F(x_3, x_2, x_1) = x_3.$$

The main results obtained by Berinde and Borcut in [14] are the following.

Theorem 1.3 (*Berinde and Borcut [14]*). *Let $(X, \preceq)$ be a partially ordered set and suppose there is a metric d on X such that (X, d) is a complete metric space. Let $F : X \times X \times X \to X$ be a continuous mapping having the mixed monotone property on X. Assume that there exist the constants $j, k, l \in [0, 1)$ with $j + k + l < 1$ for which*

$$d(F(x_1, x_2, x_3), F(y_1, y_2, y_3)) \leq jd(x_1, y_1) + kd(x_2, y_2) + ld(x_3, y_3), \tag{1.2}$$

for all $x_1 \succeq y_1, x_2 \preceq y_2, x_3 \succeq y_3$. If there exist $x_1^{(0)}, x_2^{(0)}, x_3^{(0)} \in X$ such that $x_1^{(0)} \preceq F(x_1^{(0)}, x_2^{(0)}, x_3^{(0)}), x_2^{(0)} \succeq F(x_2^{(0)}, x_1^{(0)}, x_2^{(0)})$ and $x_3^{(0)} \preceq F(x_3^{(0)}, x_2^{(0)}, x_1^{(0)})$, then there exist $x_1, x_2, x_3 \in X$ such that

$$x_1 = F(x_1, x_2, x_3), \qquad x_2 = F(x_2, x_1, x_2) \quad \text{and} \quad x_3 = F(x_3, x_2, x_1).$$

Theorem 1.4 (*Berinde and Borcut [14]*). *Let $(X, \preceq)$ be a partially ordered set and suppose there is a metric d on X such that (X, d) is a complete metric space. Assume that X has the following property:*

(i) *If a nondecreasing sequence $\{x^{(q)}\} \to x$, then $x^{(q)} \preceq x$ for all q,*
(ii) *If a non increasing sequence $\{y^{(q)}\} \to y$, then $y^{(q)} \succeq y$ for all q.*

Let $F : X \times X \times X \to X$ be a mapping having the mixed monotone property on X. Assume that there exist the constants $j, k, l \in [0, 1)$ with $j + k + l < 1$ for which (1.2) is satisfied for all $x_1 \succeq y_1, x_2 \preceq y_2, x_3 \succeq y_3$. If there exist $x_1^{(0)}, x_2^{(0)}, x_3^{(0)} \in X$ such that $x_1^{(0)} \preceq F(x_1^{(0)}, x_2^{(0)}, x_3^{(0)}), x_2^{(0)} \succeq F(x_2^{(0)}, x_1^{(0)}, x_2^{(0)})$ and $x_3^{(0)} \preceq F(x_3^{(0)}, x_2^{(0)}, x_1^{(0)})$, then there exist $x_1, x_2, x_3 \in X$ such that

$$x_1 = F(x_1, x_2, x_3), \qquad x_2 = F(x_2, x_1, x_2) \quad \text{and} \quad x_3 = F(x_3, x_2, x_1).$$

Other results related to the uniqueness of the tripled fixed point and the equality between the components of the tripled fixed point are also considered in [14].

In this paper, we extend and generalize the concept of coupled fixed point for mixed monotone mappings introduced by Bhaskar and Lakshmikantham in [12] and the concept of tripled fixed point introduced by Berinde and Borcut in [14]. We introduce the notion of fixed point of N-order for m-mixed monotone mappings and we establish existence and uniqueness fixed point theorems for such mappings satisfying a contractive condition in complete ordered metric spaces. We apply our results to the study of existence and uniqueness of solutions to some integral equations and matrix equations.