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A new algorithm for solving nearly penta-diagonal Toeplitz linear systems☆

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ABSTRACT

A new numerical algorithm for solving nearly penta-diagonal Toeplitz linear systems is presented. The algorithm is suited for implementation using Computer Algebra Systems (CASs) such as MATLAB, MACSYMA and MAPLE. Numerical examples are given in order to illustrate the efficiency of our algorithm.

1. Introduction

Linear systems of equations with nearly penta-diagonal Toeplitz coefficient matrices appear in many applications such as differential equations, parallel computing, matrix algebra, data interpolation, boundary value problems, etc [1–12].

In this paper, we consider an $n \times n$ nearly penta-diagonal Toeplitz system of linear equations given by

$$A\mathbf{x} = \mathbf{f}, \tag{1.1}$$

where

$$A = \begin{pmatrix} a_0 & a_{-1} & a_{-2} & 0 & \cdots & 0 & a_{2-n} & a_{1-n} \\ a_1 & a_0 & a_{-1} & a_{-2} & 0 & \cdots & 0 & a_{2-n} \\ a_2 & a_1 & a_0 & a_{-1} & a_{-2} & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & a_2 & a_1 & a_0 & a_{-1} & a_{-2} \\ a_{n-2} & 0 & \cdots & 0 & a_2 & a_1 & a_0 & a_{-1} \\ a_{n-1} & a_{n-2} & 0 & \cdots & 0 & a_2 & a_1 & a_0 \end{pmatrix}, \tag{1.2}$$

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$\mathbf{x} = (x_1, x_2, \ldots, x_n)^T$, $\mathbf{f} = (f_1, f_2, \ldots, f_n)^T$. Xiao-Guang Lv and Jiang Le [13] presented a fast computational algorithm for solving nearly penta-diagonal linear systems based on the use of any penta-diagonal linear solver. By using LU decomposition, Neossi Nguetchue and Abelman [14] gave a numerical algorithm to obtain the solution of nearly penta-diagonal linear systems. Recently, Xiangjian Xu [15] has proposed another efficient algorithm to solve symmetric Toeplitz penta-diagonal linear systems based on the Sherman–Morrison–Woodbury formula. In this study, we derive a numerical algorithm for solving the nearly penta-diagonal Toeplitz linear systems (1.1) and show that the total number of operations is less than those of the two algorithms in [13,14].

The rest of this paper is organized as follows: In Section 2, we describe a more efficient algorithm for solving nearly penta-diagonal Toeplitz linear systems. In Section 3, numerical examples are provided to show the performance and efficiency of our algorithm. Finally, we make some concluding remarks in Section 4.

2. Main results

We present two algorithms for solving penta-diagonal linear systems and nearly penta-diagonal Toeplitz linear systems respectively.

2.1. An algorithm for solving penta-diagonal linear systems

Consider the penta-diagonal system of linear equations $A'\mathbf{x} = \mathbf{f}$ given by

$$\begin{pmatrix} b & a_{-1} & a_{-2} & & & & \\ c & b & a_{-1} & a_{-2} & & & \\ a_2 & a_1 & a_0 & a_{-1} & a_{-2} & & \\ & \ddots & \ddots & \ddots & \ddots & \ddots & \\ & & a_2 & a_1 & a_0 & a_{-1} & a_{-2} \\ & & & a_2 & a_1 & d & e \\ & & & & a_2 & a_1 & d \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ \vdots \\ x_{n-1} \\ x_n \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ \vdots \\ \vdots \\ f_{n-1} \\ f_n \end{pmatrix}. \tag{2.1}$$

Without loss of generality, suppose that the penta-diagonal matrix A' is nonsingular and that the element $a_{-2} \neq 0$. We now introduce some notations for later use.

$$V = \begin{pmatrix} b & c & a_2 & 0 & \cdots & 0 & 0 \\ a_{-1} & b & a_1 & a_2 & 0 & \cdots & 0 \end{pmatrix}^T \in \mathbb{R}^{(n-2)\times 2},$$

$$U = \begin{pmatrix} 0 & \cdots & 0 & a_2 & a_1 & d & e \\ 0 & \cdots & 0 & 0 & a_2 & a_1 & d \end{pmatrix} \in \mathbb{R}^{2\times(n-2)},$$

$$\hat{\mathbf{f}} = (f_1, f_2, \ldots, f_{n-2})^T, \qquad \tilde{\mathbf{f}} = (f_{n-1}, f_n)^T, \qquad \mathbf{f} = \begin{pmatrix} \hat{\mathbf{f}} \\ \tilde{\mathbf{f}} \end{pmatrix},$$

$$\hat{\mathbf{x}} = (x_1, x_2)^T, \qquad \tilde{\mathbf{x}} = (x_3, x_4, \ldots, x_n)^T, \qquad \mathbf{x} = \begin{pmatrix} \hat{\mathbf{x}} \\ \tilde{\mathbf{x}} \end{pmatrix}.$$

According to the above notation, linear system (2.1) can be written in the form

$$\begin{pmatrix} V & L \\ O & U \end{pmatrix} \begin{pmatrix} \hat{\mathbf{x}} \\ \tilde{\mathbf{x}} \end{pmatrix} = \begin{pmatrix} \hat{\mathbf{f}} \\ \tilde{\mathbf{f}} \end{pmatrix}, \tag{2.2}$$

where L is a lower triangular Toeplitz matrix of size $(n-2) \times (n-2)$. Thus (2.2) is equivalent to

$$\begin{cases} V\hat{\mathbf{x}} + L\tilde{\mathbf{x}} = \hat{\mathbf{f}}, \\ U\tilde{\mathbf{x}} = \tilde{\mathbf{f}}. \end{cases} \tag{2.3}$$

Since $\det L = (a_{-2})^{n-2} \neq 0$, L is invertible. Assume that $UL^{-1}V$ is nonsingular, it is easy to deduce that

$$\begin{cases} \hat{\mathbf{x}} = (UL^{-1}V)^{-1}(UL^{-1}\hat{\mathbf{f}} - \tilde{\mathbf{f}}), \\ \tilde{\mathbf{x}} = L^{-1}(\hat{\mathbf{f}} - V\hat{\mathbf{x}}). \end{cases} \tag{2.4}$$

Proposition 2.1. *The inverse matrix of a lower triangular Toeplitz matrix is a lower triangular Toeplitz matrix.*

Proof. See [16]. □