Computers and Mathematics with Applications 66 (2013) 597–607

Contents lists available at ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

Gauss–Jacobi-type quadrature rules for fractional directional integrals☆

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ARTICLE INFO

Keywords:
Fractional directional integral
Directional derivative
Gauss–Jacobi–Lobatto
Quadrature
Fractional Laplacian

ABSTRACT

Fractional directional integrals are the extensions of the Riemann–Liouville fractional integrals from one- to multi-dimensional spaces and play an important role in extending the fractional differentiation to diverse applications. In numerical evaluation of these integrals, the weakly singular kernels often fail the conventional quadrature rules such as Newton–Cotes and Gauss–Legendre rules. It is noted that these kernels after simple transforms can be taken as the Jacobi weight functions which are related to the weight factors of Gauss–Jacobi and Gauss–Jacobi–Lobatto rules. These rules can evaluate the fractional integrals at high accuracy. Comparisons with the three typical adaptive quadrature rules are presented to illustrate the efficacy of the Gauss–Jacobi-type rules in handling weakly singular kernels of different strengths. Potential applications of the proposed rules in formulating and benchmarking new numerical schemes for generalized fractional diffusion problems are briefly discussed in the final remarking section.

1. Introduction

Recent decades have witnessed a fast growing research on the applications of fractional calculus in diverse science and engineering fields such as physics [1–3], rheology [4,5], finance [6,7], acoustics [8,9], fractal geometry [10], hydrology [11–13], etc. In particular, by replacing the second-order derivative with a derivative of fractional order $\alpha \in (1, 2]$ in the conventional advection–diffusion equation, the fractional advection–diffusion equation (FADE) appears to be a promising tool to describe solute transport in groundwater [11]. Solutions of the FADE are the Lévy-stable motions which can describe the super-diffusive flow [12]. For modeling problems in higher spatial dimensions, the fractional diffusion operator in the FADE has been extended to the weighted, fractional directional diffusion operator D_M^α from which the full range of the Lévy-stable motions can be generated [13].

The mathematical complexity of fractional derivatives often makes the analytical solutions of FADEs inaccessible [14]. Hence, the numerical solution techniques are usually resorted to. To test a numerical method for solving FADE, a reference solution with defined source term is needed. Take the 2D problem, i.e.

$$D_M^\alpha u(x, y) + f(x, y) = 0 \tag{1}$$

as an example. It is common that the solution u is pre-fixed and the source term f is numerically computed to satisfy the FADE. With f prescribed, the efficacy of the numerical method can be assessed by comparing its prediction with the

☆ Part of this work was done when the first author was visiting the University of Hong Kong as a research assistant.

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http://dx.doi.org/10.1016/j.camwa.2013.04.020

pre-fixed u. This paper presents a numerical scheme which can compute f to high accuracy for each discrete point in the computational domain.

Fractional directional integrals are involved in the definition of fractional diffusion term $D_M^\alpha u(\mathbf{x})$ where u is the solute concentration and $\mathbf{x}$ the position vector. To evaluate D_M^α, one must first calculate the fractional directional integrals. Since the fractional directional integration of even the most elementary functions may have non-closed form expression, numerical approximation is often required. The vector Grünward formula [15] is a possible choice. However, its accuracy is only $O(h)$ where h is the grid size. Integration quadrature is another alternative. Owing to the weakly singular kernel in the integrand of the fractional directional integral, the conventional quadrature rules such as Newton–Cotes, Gauss–Legendre rule, and its Kronrod refinement [16] fail to offer adequate accuracy. This constitutes a motivation to seek other quadrature rules which are more accurate and fast-convergent.

Gauss–Jacobi-type quadrature rules are potentially effective tools to evaluate fractional directional integrals. This type of rules takes the Jacobi weight function, which defines the orthogonality of the Jacobi polynomials, as the weight function. For a fractional directional integral, the weakly singular kernel $\varsigma^{\gamma-1}$ ($\gamma \in (0, 1)$) in the integrand can be transformed to $(1+\xi)^{\gamma-1}$ which becomes a special case of the Jacobi weight function $(1-\xi)^\mu(1+\xi)^\lambda$ for $\mu, \lambda > -1$. Consequently, the singularity of the integrand can be effectively removed.

Section 2 will discuss the definitions of fractional directional operators and their roles in the multi-dimensional fractional spatial operators, followed by Section 3 in which Gauss–Jacobi-type rules and their applications to fractional directional integrals are presented. In Section 4, one- and two-dimensional examples are examined and discussed. Finally, Section 5 remarks on the utility of the proposed rules in formulating and benchmarking numerical schemes for generalized fractional directional diffusion problems.

2. Fractional directional integrals and their applications

This section first introduces how the fractional directional integrals are extended from conventional n-fold definite integrals and then defines the fractional directional derivatives. In the last subsection, three typical two-dimensional fractional spatial operators are mentioned.

2.1. Directional integrals

The Cauchy formula [17] can rewrite the left-sided n-fold definite integral in a convolution form, i.e.

$$I_{a+}^n f(x) = \int_a^x \int_a^{x_n} \cdots \int_a^{x_3} \int_a^{x_2} f(x_1)dx_1 dx_2 \cdots dx_{n-1}dx_n = \frac{1}{\Gamma(n)} \int_a^x (x-\xi)^{n-1} f(\xi) d\xi \tag{2}$$

in which $f(x)$ is defined on $[a, b]$, n is a positive integer and $T(z)$ the gamma function. Similarly, the right-sided integral reads

$$I_{b-}^n f(x) = \int_x^b \int_{x_n}^b \cdots \int_{x_3}^b \int_{x_2}^b f(x_1)dx_1 dx_2 \cdots dx_{n-1}dx_n = \frac{1}{\Gamma(n)} \int_x^b (\xi-x)^{n-1} f(\xi) d\xi. \tag{3}$$

Applying the transforms $\varsigma = \pm(x-\xi)$ to formulas (2) and (3) produces

$$I_0^n f(x) = \frac{1}{\Gamma(n)} \int_0^{x-a} \varsigma^{n-1} f(x - \varsigma \cos 0) d\varsigma, \tag{4}$$

$$I_\pi^n f(x) = \frac{1}{\Gamma(n)} \int_0^{b-x} \varsigma^{n-1} f(x - \varsigma \cos \pi) d\varsigma, \quad x \in [a, b]. \tag{5}$$

The subscripts of the above integration operators "I^n" denote the integration direction $\theta \in [0, 2\pi)$. The integration operators in (4) and (5) can be generalized to rectangular domain

$$I_\theta^n g(x, y) = \frac{1}{\Gamma(n)} \int_0^{d(x,y,\theta)} \varsigma^{n-1} g(x - \varsigma \cos\theta, y - \varsigma \sin\theta) d\varsigma, \\ (x, y) \in \Omega = [a, b] \times [c, d], \qquad \theta \in [0, 2\pi). \tag{6}$$

The upper integration limit $d(x, y, \theta)$ is termed as the "backward distance" of point (x, y) to $\partial\Omega$ along the direction $\boldsymbol{\theta} = \{\cos\theta, \sin\theta\}^T$, as seen in Fig. 1. Similarly, the directional integral in three-dimensional space can be defined as

$$I_{\boldsymbol{\theta}}^n h(x, y, z) = \frac{1}{\Gamma(n)} \int_0^{d(x,y,z,\boldsymbol{\theta})} \varsigma^{n-1} h(x - \varsigma\theta_1, y - \varsigma\theta_2, z - \varsigma\theta_3) d\varsigma, \\ (x, y, z) \in \Omega = [a, b] \times [c, d] \times [e, f], \qquad \boldsymbol{\theta} = \{\theta_1, \theta_2, \theta_3\}^T, \qquad \|\boldsymbol{\theta}\|_2 = 1. \tag{7}$$