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A banded preconditioner for the two-sided, nonlinear space-fractional diffusion equation

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ABSTRACT

The method of lines is a standard method for advancing the solution of partial differential equations (PDEs) in time. In one sense, the method applies equally well to space-fractional PDEs as it does to integer-order PDEs. However, there is a significant challenge when solving space-fractional PDEs in this way, owing to the non-local nature of the fractional derivatives. Each equation in the resulting semi-discrete system involves contributions from every spatial node in the domain. This has important consequences for the efficiency of the numerical solver, especially when the system is large. First, the Jacobian matrix of the system is dense, and hence methods that avoid the need to form and factorise this matrix are preferred. Second, since the cost of evaluating the discrete equations is high, it is essential to minimise the number of evaluations required to advance the solution in time.

In this paper, we show how an effective preconditioner is essential for improving the efficiency of the method of lines for solving a quite general two-sided, nonlinear space-fractional diffusion equation. A key contribution is to show, how to construct suitable banded approximations to the system Jacobian for preconditioning purposes that permit high orders and large stepsizes to be used in the temporal integration, without requiring dense matrices to be formed. The results of numerical experiments are presented that demonstrate the effectiveness of this approach.

1. Introduction

For three hundreds years, the field of fractional calculus was mostly of interest only to pure mathematicians. More recently, there has been an explosion of interest from applied and computational mathematicians, physicists, and engineers who have contributed widely to make it the vibrant, exciting and active field it is today, with many important practical applications. As an illustration of this fact, the following books on fractional calculus, anomalous diffusion and its applications have all been published within the last five years: Baleanu et al. [1,2], Klages et al. [3], Meerschaert and Sikorskii [4], Klafter et al. [5], Mainardi [6], Tarasov [7], Sabatier et al. [8], Ortigueira [9]. For the most recent and up-to-date developments on fractional models across a wide range of disciplines, the interested reader is strongly recommended to consult these excellent works, all by eminent experts in the field.

The booming popularity of fractional models has stimulated demand for efficient solution techniques which can provide rapid insight and visualisation into solution behaviours. It is well-known that analytical solutions are available only for some special, simple (usually linear) fractional models. To solve more general fractional models (either linear or nonlinear), numerical solution techniques are preferred. During the last decade, a large amount of work has been undertaken in this

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area by many authors. Of most relevance to the present work are the finite difference spatial discretisations proposed by Meerschaert et al. [10–13]. However many other numerical techniques have been used successfully, including finite element methods (e.g. [14–16]), finite volume methods (e.g. [17–19]), spectral methods (e.g. [20,21]) and mesh-free methods (e.g. [22,23]).

A constant challenge faced by researchers in this area is the high computational expense of obtaining numerical solutions to fractional differential equations, owing to the non-local nature of fractional derivatives. The search for high-efficiency numerical methods that can significantly reduce the amount of computational time has become a new trend in the literature.

Preconditioning has been a common theme in this context, with authors seeking to reduce the cost of solving the (typically dense) linear systems or matrix function equations that arise from spatial discretisations of fractional differential equations. Yang et al. [18,15,24] have developed preconditioners based on eigenvalue deflation. Burrage et al. [16] considered both algebraic multigrid and incomplete LU preconditioning. Krylov subspace projection is a common theme amongst these works.

Methods that use high order temporal integration have also been proposed, aiming to reduce the cost of the method by reducing number of steps required. Liu et al. [25], Zhuang et al. [26] and Yang et al. [27] have used the method of lines to solve space-fractional equations, with temporal integration of up to fifth order accuracy.

Some authors have turned to fast transform methods to provide high efficiency. Wang et al. [28] showed how to exploit the Toeplitz-like structure of the coefficient matrix for the one-dimensional, two-sided, linear space-fractional diffusion equation to derive an efficient $\mathcal{O}(N \log^2 N)$ method. Wang and Wang [29] utilised fast Fourier transforms to efficiently compute the matrix–vector products in their Krylov subspace method. Pang and Sun [30] have proposed a multigrid method utilising fast Fourier transforms. Bueno-Orovio et al. [21] have considered Fourier spectral methods for space-fractional diffusion equations.

In this paper, we show how to construct a banded preconditioner that facilitates efficient solution of the two-sided, nonlinear space-fractional diffusion equation

$$\frac{\partial u}{\partial t} = \kappa(u, x, t)\left[p\frac{\partial^\alpha u}{\partial x^\alpha} + (1-p)\frac{\partial^\alpha u}{\partial(-x)^\alpha}\right] + q(u, x, t) \tag{1}$$

using the method of lines, on the finite domain $0 < x < L$ with homogeneous Dirichlet boundary conditions and initial condition $u(x, 0) = u_0(x)$. The fractional order α is assumed to satisfy $1 < \alpha \leq 2$. The function $u(x, t)$ can be interpreted as representing the concentration of a particle plume undergoing anomalous diffusion. The diffusion coefficient $\kappa(u, x, t)$ is assumed positive, and the forcing function $q(u, x, t)$ represents sources or sinks.

Meerschaert and Tadjeran [11] give the interpretation of the skewnesses $p \in 0, 1$ in terms of forward and backward jump probabilities in a stochastic model for anomalous diffusion. If $p = 0$ or $p = 1$ then (1) reduces to a one-sided space-fractional diffusion equation.

The left and right Riemann–Liouville space-fractional derivatives are defined by

$$\frac{\partial^\alpha u}{\partial x^\alpha} = \frac{1}{\Gamma(2-\alpha)}\frac{\mathrm{d}^2}{\mathrm{d}x^2}\int_0^x \frac{u(\xi, t)}{(x-\xi)^{\alpha-1}}\,\mathrm{d}\xi \tag{2}$$

and

$$\frac{\partial^\alpha u}{\partial(-x)^\alpha} = \frac{1}{\Gamma(2-\alpha)}\frac{\mathrm{d}^2}{\mathrm{d}x^2}\int_x^L \frac{u(\xi, t)}{(\xi-x)^{\alpha-1}}\,\mathrm{d}\xi. \tag{3}$$

In this paper, we use Meerschaert and Tadjeran's [11] finite difference scheme for Eq. (1) within a method of lines framework and Krylov subspace iterative methods to solve the resulting linear systems.

We demonstrate, how an effective preconditioner is crucial for improving the efficiency this approach. In particular, we show how to construct suitable banded approximations to the system Jacobian for preconditioning purposes, which permits high orders and large stepsizes to be used in the temporal integration, without requiring dense matrices to be formed and factorised. This allows for the solution to be obtained using many more spatial nodes than would be possible if using direct Jacobian factorisation.

It is interesting to note that, Wang et al. [28] have also exploited the approximately banded nature of the Jacobian matrix in their work. However they take a different approach from what we propose here. Theirs is a direct method, based on a splitting of the matrix into two parts: a banded portion and the remainder. They do not consider the nonlinear problem or high order temporal integration, as in this work.

The remainder of the paper is arranged as follows. In Section 2 we present a finite difference spatial discretisation of (1) and show how it leads to a system of time ordinary differential equations. In Section 3 we summarise the method of backward differentiation formulae for integrating initial value problems. The role of preconditioning is highlighted. In Section 4 we derive a banded preconditioner that is suitable for the nonlinear two-sided space-fractional diffusion equation, and show how to construct it efficiently. In Section 5 we present the results of numerical experiments that confirm that the preconditioner allows for the efficient solution of Eq. (1) using the method of lines. We draw our conclusions in Section 6.