Computers and Mathematics with Applications 66 (2013) 717–727

Contents lists available at ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

A family of Adams exponential integrators for fractional linear systems ☆

Roberto Garrappa *

Università degli Studi di Bari, Dipartimento di Matematica, Via E. Orabona n.4, 70125 Bari, Italy

ARTICLE INFO

Keywords:
Fractional differential equations
Numerical methods
Exponential integrators
Adams methods
Mittag-Leffler function

ABSTRACT

The numerical solution of linear time-invariant systems of fractional order is investigated. We construct a family of exponential integrators of Adams type possessing good convergence and stability properties. The methods are devised in order to keep at a suitable level, the computational effort necessary to solve problems of large size. Numerical experiments are provided to validate the theoretical results; the effectiveness of the proposed approach is tested and compared to some other classical methods.

1. Introduction

In several applications there are frequently involved linear time-invariant (LTI) systems of fractional order in the form

$$\begin{aligned} &\frac{d^\alpha}{dt^\alpha}x(t) = Ax(t) + bu(t) \\ &y(t) = c^T x(t), \end{aligned} \tag{1}$$

where α is any real positive number, $A \in \mathbb{R}^{N\times N}$, $b \in \mathbb{R}^{N\times N_i}$ and $c \in \mathbb{R}^{N\times N_o}$ are constant and $x(t)$, $u(t)$ and $y(t)$ are vector-valued functions mapping the interval $[0, +\infty)$ respectively into $\mathbb{R}^N$, $\mathbb{R}^{N_i}$ and $\mathbb{R}^{N_o}$.

Throughout the paper, the fractional derivative, with respect to the origin, is introduced according to the Caputo's definition

$$\frac{d^\alpha}{dt^\alpha}x(t) = \frac{1}{\Gamma(\lceil\alpha\rceil - \alpha)} \int_0^t \frac{x^{(\lceil\alpha\rceil)}(\tau)}{(t-\tau)^{\alpha+1-\lceil\alpha\rceil}} d\tau,$$

where $\Gamma(\cdot)$ is the Euler gamma function, $\lceil\alpha\rceil$ is the smallest integer such that $\alpha < \lceil\alpha\rceil$ and $x^{(\lceil\alpha\rceil)}$ denotes the standard derivative of integer order $\lceil\alpha\rceil$. This approach is the most natural in real-life applications since it allows to couple system (1) with classical initial conditions of Cauchy type

$$x^{(\nu)}(0) = x_{0,\nu}, \quad \nu = 0, \dots, \lceil\alpha\rceil - 1.$$

We refer, for instance, to [1–3] for basic materials on fractional calculus.

Systems of this kind are frequently encountered in control theory; in this case $x(t)$ represents the state or pseudo-state variables and $u(t)$ and $y(t)$ denote respectively the input and output of the system. We are interested in problems with a

☆ This work is supported by the project "High-performance robust fractional controllers for applications in industry and mechatronics", which is part of the national project PRIN 2009F4NZJP "Non integer order systems in modeling and control", funded by the Italian Ministry of University and Research (MIUR).

* Tel.: +39 0805442685.
E-mail addresses: garrappa@dm.uniba.it, roberto.garrappa@uniba.it.

0898-1221/$ – see front matter
doi:10.1016/j.camwa.2013.01.022

highly internal complexity in which not only N is very large but it is also several orders of magnitude larger than N_i and N_o. The case $N_i = N_o = 1$ is also of practical interest. For instance, models with a very high degree of complexity have been proposed in the fields of electrical networks, power systems and smart grids [4]. Very often, problems of this kind naturally originate from models described by partial differential equations (PDEs) with fractional-time derivatives after the discretization of 2D and 3D spatial derivatives by some suitable techniques (usually finite elements or finite differences); in these instances the number of internal states can vary from 10^3 to 10^6 or even 10^9.

The numerical solution of systems of fractional differential equations (FDEs), especially of very large size, is usually very demanding. Indeed, derivative operators of non-integer order are non-local operators and, consequently, the time-step integration involves, at each step, the use of all the previously evaluated values, with an increasing computational effort as the integration process moves forward. Furthermore, with *stiff* problems some restrictive bounds apply to the step-size in order to avoid errors for propagating very quickly.

Because of the combined effects of long-term memory and use of small step-sizes, the computation can become prohibitive for practical purposes.

By chance, in most applications the main interest is not devoted to study (and compute) the evolution of the state $x(t)$, but rather to measure the response of the output $y(t)$ under the effects of a certain excitation $u(t)$. Especially with systems in which N_i and N_o are reasonably smaller than N, our idea is that some advantages can be taken by directly evaluating the output $y(t)$ from the input $u(t)$ without explicitly calculating $x(t)$.

In this paper we introduce a new family of methods of Adams type specifically designed for numerically solving fractional LTI systems in order to achieve the following two main goals: (1) providing highly accurate solutions with stable behavior without imposing severe restrictions on the step-size even in the presence of stiff problems; (2) keeping the computational effort at a reasonable level by performing most of the computation in a space of size proportional to N_i and N_o instead of N.

The framework in which we will be operating is the exponential integrators. This is a class of numerical methods which has been widely used and investigated for ordinary differential equations (ODEs); their use within fractional calculus is however a rather unexplored area. To achieve the above mentioned goals, exponential integrators will be combined with Krylov subspace methods and rational approximations.

This paper is organized as follows: in Section 2 we review some basic facts about exponential integrators for ODEs and Adams methods for FDEs. Moreover we introduce a variation of constant formula for FDEs in order to describe in details, in Section 3, the way in which exponential integrators of Adams type are generalized to FDEs; convergence properties are also studied. In Section 4 we address the problem of evaluating, in an efficient way, the convolution weights (which are special functions with matrix arguments) and we discuss some important issues related to the treatment of large size problems. Finally, in Section 5 we present some numerical experiments to validate the theoretical findings and show the effectiveness of the proposed approach by means of some comparisons with other existing methods.

2. Preliminaries

For the sake of clarity and completeness, we briefly review some basic material which will be used later in the paper. We will include in this section just the essential notions necessary for the subsequent analysis and we will provide some references for the related topics.

2.1. Exponential integrators for ODEs

Exponential integrators are a class of powerful methods specifically designed for solving semilinear ODEs; basically, the linear term is separated and solved by a matrix exponential and a time-stepping technique is applied to the nonlinear term. The key tool in this process is the classical variation-of-constant formula for ODEs.

The main advantage is that once the linear term (which usually is stiff) has been solved exactly, the restrictions on the step-size due to stability requirements are greatly relaxed. We do not discuss this technique in more details since the way in which exponential integrators are devised will be clear when, in the next section, we will face with their generalization to FDEs; we just refer to the review in [5] and to [6–8] for specific results on exponential integrators of Adams type for ODEs.

2.2. Adams methods for FDEs

A classical approach for the time simulation of the FDE in (1) starts from the integral representation of the true solution

$$x(t) = \sum_{\nu=0}^{\lceil\alpha\rceil-1} \frac{t^\nu}{\nu!} x_{0,\nu} + \int_0^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} \big(Ax(\tau) + bu(\tau)\big) d\tau .$$

On a uniform grid-mesh with constant spacing $h > 0$, the vector field $Ax(t) + bu(t)$ is approximated by a suitable piecewise polynomial and the resulting integrals are exactly evaluated. This technique, which is a generalization of classical Adams multistep methods for ODEs, usually goes under the name of Product Integration (PI) rules [9].