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The Adomian decomposition method with convergence acceleration techniques for nonlinear fractional differential equations

Jun-Sheng Duan [a,*], Temuer Chaolu [b], Randolph Rach [c], Lei Lu [a]

[a] School of Sciences, Shanghai Institute of Technology, Shanghai 201418, PR China

[b] College of Sciences and Arts, Shanghai Maritime University, Shanghai 200135, PR China

[c] 316 South Maple Street, Hartford, MI 49057-1225, USA

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ABSTRACT

In this paper, we present the Adomian decomposition method and its modifications combined with convergence acceleration techniques, such as the diagonal Padé approximants and the iterated Shanks transforms, to solve nonlinear fractional ordinary differential equations. Two nonlinear numeric examples demonstrate that either the diagonal Padé approximants or the iterated Shanks transforms can efficiently extend the effective convergence region of the decomposition series solution.

1. Introduction

In recent decades, fractional differential equations, a class of integro-differential equations with singularities, have been extensively applied in various fields of science and engineering, such as viscoelasticity, anomalous diffusion, biology, control theory, etc. [1–5].

Several analytical or numerical methods have been previously proposed to solve fractional differential equations, such as various integral transform methods [2–7] for the linear case, the Adomian decomposition method (ADM) [8–13], variational iteration method [14–16] and orthogonal polynomial method [2,17] for the nonlinear case, and various numerical methods [2–5,18–21] as well as other methods [2–5,22–24].

We say that $f(t)$ is a function of class $\mathfrak{C}$, if $f(t)$ is piecewise continuous on $(0, +\infty)$ and integrable on any finite subinterval of $(0, +\infty)$. Let $f(t)$ be a function of class $\mathfrak{C}$, then the Riemann–Liouville fractional integral of $f(t)$ of order β is defined as

$$J_t^{\beta} f(t) = \int_0^t \frac{(t-\tau)^{\beta-1}}{\Gamma(\beta)} f(\tau) d\tau, \quad \beta > 0, \tag{1}$$

where $\Gamma(\cdot)$ is Euler's gamma function. We define $J_t^0 f(t) = f(t)$. The fractional integral satisfies the following equality,

$$J_t^{\nu} t^{\mu} = \frac{\Gamma(\mu+1)}{\Gamma(\mu+\nu+1)} t^{\mu+\nu}, \quad \nu \geq 0, \mu > -1. \tag{2}$$

Let $f(t)$ be a function of class $\mathfrak{C}$ and α be a positive real number satisfying $m-1 < \alpha \leq m$, $m \in \mathbb{N}^+$. Then the Riemann–Liouville fractional derivative of $f(t)$ of order α, when it exists, is defined as

$$\mathbf{D}_t^{\alpha} f(t) = \frac{d^m}{dt^m} \left(J_t^{m-\alpha} f(t) \right), \quad t > 0. \tag{3}$$

* Corresponding author. Tel.: +86 15002149381; fax: +86 60873193.
E-mail addresses: duanjssdu@sina.com, duanjs@sit.edu.cn (J.-S. Duan), tmchaolu@shmtu.edu.cn (T. Chaolu), tapstrike@triton.net (R. Rach), lulei1698@163.com (L. Lu).

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Let α be a positive real number, such that $m-1<\alpha\le m$, $m\in\mathbb{N}^+$, and $f^{(m)}(t)$ exist and be a function of class $\mathfrak{C}$. Then the Caputo fractional derivative of $f(t)$ of order α is defined as

$$D_t^\alpha f(t)=J_t^{m-\alpha}f^{(m)}(t),\quad t>0. \tag{4}$$

For the Caputo fractional derivative of a polynomial function, the following equality holds

$$D_t^\alpha(a_0t^r+a_1t^{r-1}+\cdots+a_r)=0,\quad m-1<\alpha\le m, r\le m-1. \tag{5}$$

Moreover, the α-order integral of the α-order Caputo fractional derivative satisfies

$$J_t^\alpha D_t^\alpha f(t)=f(t)-\sum_{k=0}^{m-1}f^{(k)}(0^+)\frac{t^k}{k!},\quad m-1<\alpha\le m. \tag{6}$$

For the power function t^μ, $\mu>0$, if $0\le m-1<\alpha\le m<\mu+1$, then we have

$$D_t^\alpha t^\mu=\frac{\Gamma(\mu+1)}{\Gamma(\mu-\alpha+1)}t^{\mu-\alpha},\quad t>0. \tag{7}$$

The Caputo fractional derivative and the Riemann–Liouville fractional derivative satisfy the following relation,

$$D_t^\alpha f(t)=\mathbf{D}_t^\alpha\left[f(t)-\sum_{k=0}^{m-1}\frac{t^k}{k!}f^{(k)}(0^+)\right]. \tag{8}$$

The two fractional derivatives have their respective advantageous features. The model of the initial value problems (IVPs) involving the Caputo fractional derivatives has a similar form compared to the classical models. We use the Caputo fractional derivatives to form the fractional differential equations in this work.

In next section, we introduce the ADM and its modifications combined with two common convergence acceleration methods. Two numeric examples demonstrate the efficacy of our combined techniques in Section 3.

2. The ADM with convergence acceleration techniques

2.1. The ADM and its modifications

The ADM [25–27] is a systematic method to solve both linear and nonlinear functional equations. We consider the IVP for the nonlinear fractional ordinary differential equation (ODE),

$$D_t^\lambda u(t)+f(u(t))=g(t),\quad 1<\lambda\le 2, \tag{9}$$

$$u(0)=C_0,\qquad u'(0)=C_1, \tag{10}$$

where f is an analytic nonlinear function and $g(t)$ is the system input.

Applying the fractional integral operator J_t^λ to both sides of Eq. (9) yields

$$u(t)=C_0+C_1t+J_t^\lambda g(t)-J_t^\lambda f(u(t)). \tag{11}$$

We decompose the solution as $u(t)=\sum_{n=0}^\infty u_n$, and then decompose the analytic nonlinearity $Nu=f(u(t))$ into the series of the Adomian polynomials

$$f(u(t))=\sum_{n=0}^\infty A_n, \tag{12}$$

where the Adomian polynomials $A_n=A_n(u_0,u_1,\dots,u_n)$ are defined by the formula [25]

$$A_n=\frac{1}{n!}\frac{d^n}{d\lambda^n}f\left(\sum_{k=0}^\infty u_k\lambda^k\right)\Bigg|_{\lambda=0},\quad n\ge 0. \tag{13}$$

Several algorithms such as in [25,28,27,29] for symbolic programming have since been devised to efficiently generate the Adomian polynomials quickly and to high orders. New, more efficient algorithms and subroutines in MATHEMATICA for rapid generation of the Adomian polynomials have been provided by Duan in [30–32]. We list Duan's Corollary 3 algorithm [32] as follows:

$$C_n^1=u_n,\quad n\ge 1,$$

$$C_n^k=\frac{1}{n}\sum_{j=0}^{n-k}(j+1)u_{j+1}C_{n-1-j}^{k-1},\quad 2\le k\le n, \tag{14}$$

$$A_n=\sum_{k=1}^n C_n^k f^{(k)}(u_0),\quad n\ge 1.$$