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Propagation speed of the maximum of the fundamental solution to the fractional diffusion–wave equation

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ABSTRACT

In this paper, the one-dimensional time-fractional diffusion–wave equation with the fractional derivative of order $\alpha,\ 1 < \alpha < 2$, is revisited. This equation interpolates between the diffusion and the wave equations that behave quite differently regarding their response to a localized disturbance: whereas the diffusion equation describes a process, where a disturbance spreads infinitely fast, the propagation speed of the disturbance is a constant for the wave equation. For the time-fractional diffusion–wave equation, the propagation speed of a disturbance is infinite, but its fundamental solution possesses a maximum that disperses with a finite speed. In this paper, the fundamental solution of the Cauchy problem for the time-fractional diffusion–wave equation, its maximum location, maximum value, and other important characteristics are investigated in detail. To illustrate analytical formulas, results of numerical calculations and plots are presented. Numerical algorithms and programs used to produce plots are discussed.

1. Introduction

Evolution equations related to phenomena intermediate between diffusion and wave propagation have attracted the attention of a number of researchers since the 1980s. This kind of phenomenon is known to occur in viscoelastic media that combine the characteristics of solid-like materials that exhibit wave propagation and fluid-like materials that support diffusion processes. In particular, analysis and results presented by Pipkin in [1] and Kreis and Pipkin [2] should be mentioned. Being unaware of an interpretation of evolution equations by means of fractional calculus, these authors still could provide an interesting example of the relevance of the intermediate phenomena for models in continuum mechanics.

Nowadays it is well recognized that evolution equations can be interpreted as differential equations of fractional order at the time when some hereditary mechanisms of power-law type are present in diffusion or wave phenomena. This has been shown for example in [3–5] and more recently in [6,7], where propagation of pulses in linear lossy media governed by constitutive equations of fractional order has been revisited.

For analysis of the evolution equations of the type mentioned above, methods and tools of fractional calculus, integral transforms, and higher transcendental functions have been employed in the pioneering papers [8–13], and in the book [14]. We also mention the papers by Mainardi [15–18], where fundamental solutions of the evolution equations related to phenomena intermediate between diffusion and wave propagation have been expressed in terms of some auxiliary functions of the Wright type that sometimes are referred to as Mainardi functions, see i.e. [19–21]. These functions as well as some

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techniques and methods of fractional calculus, integral transforms, and higher transcendental functions will be used in our analysis.

It is well known that diffusion and wave equations behave quite differently regarding their response to a localized disturbance: whereas the diffusion equation describes a process, where a disturbance spreads infinitely fast, the propagation speed of the disturbance is constant for the wave equation. In a certain sense, the time-fractional diffusion–wave equation interpolates between these two different responses. On the one hand, the support of the solution to this equation is not compact on the real line for each $t > 0$ for a non-negative disturbance that is not identically equal to zero, i.e. its response to a localized disturbance spreads infinitely fast (see [8]). On the other hand, the fundamental solution of the time-fractional diffusion–wave equation possesses a maximum that disperses with a finite speed similar to the behavior of the fundamental solution of the wave equation. The problem to describe the location of the maximum of the fundamental solution of the Cauchy problem for the one-dimensional time-fractional diffusion–wave equation of order α, $1 < \alpha < 2$, was considered for the first time by Fujita in [8]. Fujita proved that the fundamental solution takes its maximum at the point $x_* = \pm c_\alpha t^{\alpha/2}$ for each $t > 0$, where $c_\alpha > 0$ is a constant determined by α. Recently, another proof of this formula for the maximum location along with numerical results for the constant c_α for $1 < \alpha < 2$ were presented by Povstenko in [22]. In this paper, we provide an extension and consolidation of these results along with some new analytical formulas, numerical algorithms, and pictures.

The rest of the paper is organized as follows: in the 2nd section, problem formulation and some analytical results are given. Here we revisit the results of Fujita and Povstenko and give some new insights into the problem. Especially the role of the symmetry group of scaling transformations of the time-fractional diffusion–wave equation in the maximum propagation problem is emphasized. We derive a new formula for the maximum value of the Green function for the Cauchy problem for the time-fractional diffusion–wave equation. A new characteristic of the time-fractional diffusion–wave equation – the product of the maximum location of its fundamental solution and its maximum value – is introduced. For a fixed value of α, $1 \le \alpha \le 2$, this product is a constant for all $t > 0$ that depends only on α. The product is equal to zero for the diffusion equation and to infinity for the wave equation, whereas it is finite, positive, and lying between these extreme values for the time-fractional diffusion–wave equation that justifies the fact that the time-fractional diffusion–wave equation interpolates between the diffusion and the wave equations. The 3rd section is devoted to a presentation of the numerical algorithms used to calculate the fundamental solution and its important characteristics including the location of its maximum, its propagation speed, and the maximum value. Results of numerical calculations and plots are presented and discussed in detail.

2. Problem formulation and analytical results

This section is devoted to the problem formulation and some important analytical results. In particular, several representations of the fundamental solution of the Cauchy problem for the time-fractional diffusion–wave equation in the form of series and integrals are given. These representations are used to derive explicit formulas for the maximum location, maximum value, and the propagation speed of the maximum point. Besides, we give a new proof of the fact that a response of the time-fractional diffusion–wave equation to a localized disturbance spreads infinitely fast like in the case of the diffusion equation.

2.1. Problem formulation

In this paper, we deal with the family of evolution equations obtained from the standard diffusion equation (or the D'Alembert wave equation) by replacing the first-order (or the second-order) time derivative by a fractional derivative of order α with $1 \le \alpha \le 2$, namely

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{\partial^2 u}{\partial x^2}, \tag{1}$$

where $x \in \mathbb{R}$, $t \in \mathbb{R}^+$ denote the space and time variables, respectively.

In (1), $u = u(x, t)$ represents the response field variable and the fractional derivative of order α, $n - 1 < \alpha < n$, $n \in \mathbb{N}$ is defined in the Caputo sense:

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} \frac{\partial^n u(\tau)}{\partial \tau^n}\, d\tau, \tag{2}$$

where Γ denotes the Gamma function. For $\alpha = n$, $n \in \mathbb{N}$, the Caputo fractional derivative is defined as the standard derivative of order n.

In order to guarantee existence and uniqueness of a solution, we must add to (1) some initial and boundary conditions. Denoting by $f(x)$, $x \in \mathbb{R}$ and $h(t)$, $t \in \mathbb{R}^+$ sufficiently well-behaved functions, the Cauchy problem for the time-fractional diffusion–wave equation with $1 \le \alpha \le 2$ is formulated as follows:

$$\begin{cases} u(x,0) = f(x), & -\infty < x < +\infty; \\ u(\mp\infty, t) = 0, & t > 0. \end{cases} \tag{3}$$

If $1 < \alpha \le 2$, we must add to (3) the initial value of the first time derivative of the field variable, $u_t(x, 0)$, since in this case the Caputo fractional derivative is expressed in terms of the second order time derivative. To ensure continuous