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A mathematical model for simulation of a water table profile between two parallel subsurface drains using fractional derivatives

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ABSTRACT

By considering the initial and boundary conditions corresponding to parallel subsurface drains, the linear form of a one-dimensional fractional Boussinesq equation was solved and an analytical mathematical model was developed to predict the water table profile between two parallel subsurface drains. The developed model is a generalization of the Glover–Dumm's mathematical model. As a result, the new model is applicable for both homogeneous and heterogeneous soils. It considers the degree of heterogeneity of soil as a determinable parameter. This parameter was called the heterogeneity index. The laboratory and field tests were conducted to study the performance of the proposed mathematical model in a homogenous soil and in an agricultural soil. The optimal values of parameters of the fractional model developed in this study and Glover–Dumm's model were estimated using the inverse problem method. In the proposed inverse model, a bees algorithm (BA) was used. The results showed that in the homogenous soil, the heterogeneity index was nearly equal to 2 and therefore, the developed mathematical model reduced to the Glover–Dumm's mathematical model. The heterogeneity index of the experimental field soil considered was equal to 1.04; hence, this soil was classified as a very heterogeneous soil. In the experimental field soil, the proposed mathematical model better represented the water table profile between two parallel subsurface drains than the Glover–Dumm's mathematical model. Therefore, it appears that the proposed fractional model presented is a highly general and effective method to estimate the water table profile between two parallel subsurface drains, and the scale effects are robustly reflected by the introduced heterogeneity index.

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1. Introduction

Predicting fluctuations of a water table is very important from an agricultural and environmental perspective. Groundwater flow in an unconfined aquifer can be simulated using the Boussinesq equation. The Boussinesq equation is given by [1]:

$$\frac{\partial}{\partial x}\left(K_x h\frac{\partial h}{\partial x}\right)+\frac{\partial}{\partial y}\left(K_y h\frac{\partial h}{\partial y}\right)+N=S_y\frac{\partial h}{\partial t}, \tag{1}$$

where K_x is the saturated hydraulic conductivity in the x direction (L/T), K_y is the saturated hydraulic conductivity in the y direction (L/T), h is the hydraulic head (L), S_y is the specific yield (dimensionless), and N is the recharge rate or discharge rate (L/T).

Further applications of this equation to the most non-steady drainage equations were reported (e.g., [2–7]). For more details about Boussinesq and its limitations we refer to [8,9].

The fractional derivatives were used extensively in the last years to improve the existing models describing the porous media.

The Caputo space-fractional derivative of order α for a and $x > a$ is defined as follows [10–12]:

$$\left(D_a^{\alpha}f\right)(x,t)=\frac{1}{\Gamma(m-\alpha)}\cdot\int_a^x\frac{f^{(m)}(y,t)}{(x-y)^{\alpha-m+1}}dy,\quad m-1<\alpha\leq m, \tag{2}$$

where m is the smallest integer greater than α, and $\Gamma(\cdot)$ is the Gamma function. The fractional Taylor series is a generalization of the Taylor series. The fractional Taylor series of $f(x)$ at the point $x+\Delta x$ in the Caputo sense is defined as [12]:

$$f(x+\Delta x)=f(x)+\frac{\partial^{\alpha}f(x)}{\partial x^{\alpha}}\cdot\frac{\Delta x^{\alpha}}{\Gamma(1+\alpha)}+\frac{\partial^{\alpha}}{\partial x^{\alpha}}\left(\frac{\partial^{\alpha}f(x)}{\partial x^{\alpha}}\right)\cdot\frac{\Delta x^{2\alpha}}{\Gamma(1+2\alpha)}+\cdots, \tag{3}$$

where $0 < \alpha \leq 1$.

One of the most important properties of fractional derivatives, in contrast to integer-order derivatives, is their property of non-locality (see for more details Refs. [13–19]).

Recently, the fractional derivatives have been used in many diverse fields of science and engineering. In hydrogeology, they appear in the simulation of groundwater flow and solute transport in groundwater (see for example Refs. [17,20–24,18] and the references therein). Wheatcraft and Meerschaert [19] suggested a fractional mass conservation by assuming power-law changes of flux through the control volume and using a fractional Taylor series. Barker [25] proposed a generalized radial flow (GRF) model for hydraulic tests in fractured formations. Barker [25] took into account the flow dimension as a parameter that includes arbitrary values in the range [1, 3]. He [26] presented a generalized Darcy's law using fractional derivatives:

$$v=-K_x\frac{\partial^{\eta}h}{\partial x^{\eta}}\quad 0<\eta\leq 1. \tag{4}$$

He [26] did not define the parameters of Eq. (4). One can explain them as follow:

v is the velocity of groundwater flow (L/T), K_x is the fractional hydraulic conductivity (L^{η}/T), η $(\eta \in (0,1])$ is the order of differentiation that indicates the degree of heterogeneity (dimensionless), and h is the hydraulic head (L).

Cloot and Botha [27] used the generalized Darcy's law and the law of conservation of mass to derive a new equation for groundwater radial flow. Gehlhausen [28] evaluated a fractional Theis solution with aquifer test data from the Campus Test Site at the University of the Free State, South Africa. The results indicated the necessity of using the fractional Theis equation for one-dimensional flow.

In this paper, an analytical mathematical model is derived by solving the linear fractional Boussinesq equation for the initial and boundary conditions corresponding to parallel subsurface drains. The parameters of the derived mathematical model and Glover–Dumm's mathematical model are estimated using the inverse problem method. In addition, the accuracy of the obtained mathematical model for simulation of the water table profile between two subsurface drains in a homogenous soil and in an agricultural soil is studied and compared with the accuracy of Glover–Dumm's model.

2. Theoretical development

2.1. The model

To develop the fractional Boussinesq equation, consider the fluid mass conservation for the control volume bounded by vertical surfaces at x, $x+\Delta x$, y and $y+\Delta y$ as shown in Fig. 1 [29].

If the variation of h relative to the value of h is infinitesimal, one can consider that the average saturated thickness is equal to a constant value and derive a linear fractional Boussinesq equation [29]:

$$\kappa_x\cdot D\frac{\partial^{\nu}h}{\partial x^{\nu}}+\kappa_y\cdot D\frac{\partial^{\mu}h}{\partial y^{\mu}}+N=S_y\frac{\partial h}{\partial t}. \tag{5}$$