Available online at www.sciencedirect.com
ScienceDirect
Computers and Mathematics with Applications 54 (2007) 1367–1379
An International Journal
computers & mathematics with applications
www.elsevier.com/locate/camwa

Sturm–Liouville problems with discontinuities at two points

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Received 9 December 2005; received in revised form 24 May 2006; accepted 30 May 2006

Abstract

In this paper we extend some spectral properties of regular Sturm–Liouville problems to those which consist of a Sturm–Liouville equation with piecewise continuous potentials together with eigenparameter-dependent boundary conditions and four supplementary transmission conditions. By modifying some techniques of [C.T. Fulton, Two-point boundary value problems with eigenvalue parameter contained in the boundary conditions, Proc. Roy. Soc. Edinburgh Sect. A 77 (1977) 293–308; E. Tunç, O.Sh. Muhtarov, Fundamental solutions and eigenvalues of one boundary-value problem with transmission conditions, Appl. Math. Comput. 157 (2004) 347–355; O.Sh. Mukhtarov, E. Tunç, Eigenvalue problems for Sturm–Liouville equations with transmission conditions, Israel J. Math. 144 (2004) 367–380] and [O.Sh. Mukhtarov, M. Kadakal, F.Ş. Muhtarov, Eigenvalues and normalized eigenfunctions of discontinuous Sturm–Liouville problem with transmission conditions, Rep. Math. Phys. 54 (2004) 41–56], we give an operator-theoretic formulation for the considered problem and obtain asymptotic formulae for the eigenvalues and eigenfunctions.

Keywords: Sturm–Liouville problems; Transmission conditions; The asymptotic of eigenvalues and eigenfunctions; Discontinuous boundary-value problems; Eigenparameter-dependent boundary conditions

1. Introduction

Sturmian theory is one of the most extensively developing fields in theoretical and applied mathematics. Particularly, there has been increasing interest in the spectral analysis of boundary value problems with eigenvalue-dependent boundary conditions. There are quite substantial literatures on such problems. Here we mention the results of [1,5–12] and the corresponding references cited therein.

Basically, boundary-value problems with continuous coefficients at the highest derivative of the equation have been investigated. Note that discontinuous Sturm–Liouville problems with eigen-dependent boundary conditions and with two supplementary transmission conditions at the point(s) of discontinuity have been investigated in [2–4,13,14]. In this paper, we shall consider discontinuous eigenvalue problem which consist of the differential equation

$$\tau u := -u'' + q(x)u = \lambda u \tag{1.1}$$

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0898-1221/$ - see front matter © 2007 Published by Elsevier Ltd
doi:10.1016/j.camwa.2006.05.032

on $[a,\xi_1)\cup(\xi_1,\xi_2)\cup(\xi_2,b]$, with boundary condition at $x=a$

$$L_1u := \alpha_1 u(a) + \alpha_2 u'(a) = 0, \tag{1.2}$$

with the four transmission conditions at the points of discontinuity $x=\xi_1$ and $x=\xi_2$,

$$L_2u := \gamma_1 u(\xi_1-0) - \delta_1 u(\xi_1+0) = 0 \tag{1.3}$$

$$L_3u := \gamma_1' u'(\xi_1-0) - \delta_1' u'(\xi_1+0) = 0 \tag{1.4}$$

$$L_4u := \gamma_2 u(\xi_2-0) - \delta_2 u(\xi_2+0) = 0 \tag{1.5}$$

$$L_5u := \gamma_2' u'(\xi_2-0) - \delta_2' u'(\xi_2+0) = 0 \tag{1.6}$$

and the eigen-dependent boundary condition at $x=b$

$$L_6(\lambda)u := \lambda\left[\beta_1' u(b) - \beta_2' u'(b)\right] + \left[\beta_1 u(b) - \beta_2 u'(b)\right] = 0 \tag{1.7}$$

where $q(x)$ is a given real-valued function continuous in $[a,\xi_1]$, $[\xi_1,\xi_2]$ and $[\xi_2,b]$ (that is, continuous in $[a,\xi_1)$, (ξ_1,ξ_2) and $(\xi_2,b]$ and has finite limits $q(\xi_1\pm) := \lim_{x\to\xi_1\pm} q(x)$, $q(\xi_2\pm) := \lim_{x\to\xi_2\pm} q(x)$); λ is a complex eigenvalue parameter; the coefficients of the boundary and transmission conditions are real numbers. We assume $|\alpha_1|+|\alpha_2|\neq 0$, $|\gamma_i|+|\delta_i|\neq 0$, $\left|\gamma_i'\right|+\left|\delta_i'\right|\neq 0\,(i=1,2)$ and $\rho = \begin{vmatrix} \beta_1' & \beta_1 \\ \beta_2' & \beta_2 \end{vmatrix} > 0$. In contrast to previous works, the eigenfunctions of this problem may have discontinuities.

Note that problems of such a type arise, as a rule, in the theory of heat and mass transfer problems, and in a varied assortment of physical transfer problems. (See [1,8] and [15] and corresponding references cited therein for various physical applications.)

2. Preliminaries

For convenience let us introduce the following notations:

$$\Omega_1 := [a,\xi_1],\quad \Omega_2 := [\xi_1,\xi_2],\quad \Omega_3 := [\xi_2,b],\quad u_{(1)}(x) := \begin{cases} u(x) & x\in[a,\xi_1) \\ \lim\limits_{x\to\xi_1-} u(x) & x=\xi_1, \end{cases}$$

$$u_{(2)}(x) := \begin{cases} u(x) & x\in(\xi_1,\xi_2) \\ \lim\limits_{x\to\xi_1+} u(x) & x=\xi_1, \end{cases}\qquad u_{(3)}(x) := \begin{cases} u(x) & x\in(\xi_1,\xi_2) \\ \lim\limits_{x\to\xi_2-} u(x) & x=\xi_2, \end{cases}$$

$$u_{(4)}(x) := \begin{cases} u(x) & x\in(\xi_2,b] \\ \lim\limits_{x\to\xi_2+} u(x) & x=\xi_2 \end{cases}$$

$$(u)_\beta := \lim_{x\to b}\left(\beta_1 u(x) - \beta_2 u'(x)\right),\quad (u)'_\beta := \lim_{x\to b}\left(\beta_1' u(x) - \beta_2' u'(x)\right),\quad \tilde{u}(x) = \begin{cases} u(x), & x\in[a,b) \\ (u)'_\beta, & x=b. \end{cases}$$

Note that everywhere below, we shall assume that $\gamma_i\gamma_i'\delta_i\delta_i' > 0\,(i=1,2)$, and for the Lebesque measurable subsets $M\subset[a,\xi_1)\cup(\xi_1,\xi_2)\cup(\xi_2,b]$ with Lebesque measure $\mu_L(M)$, we shall define a new positive measure $\mu_\rho(M)$ by

$$\mu_\rho(M) := \frac{\gamma_1\gamma_1'}{\delta_1\delta_1'}\mu_L(M\cap[a,\xi_1)) + \mu_L(M\cap(\xi_1,\xi_2)) + \frac{\delta_2\delta_2'}{\gamma_2\gamma_2'}\mu_L(M\cap(\xi_2,b]) + \frac{\delta_2\delta_2'}{\gamma_2\gamma_2'}\frac{b(M)}{\rho}$$

where

$$b(M) := \begin{cases} 0 & \text{if } b\notin M \\ 1 & \text{if } b\in M. \end{cases}$$

Let $\langle\cdot,\cdot\rangle_{H_\rho}$ denote the scalar product in the Hilbert space $H_\rho := L^2\left([a,b];\mu_\rho\right)$. In this space, we define a linear operator A by the domain of definition