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Some integral inequalities for functions with $(n-1)$st derivatives of bounded variation

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Abstract

In this paper, we generalize Cerone's results, and a unified treatment of error estimates for a general inequality satisfying $f^{(n-1)}$ being of bounded variation is presented. We derive the estimates for the remainder terms of the mid-point, trapezoid, and Simpson formulas. All constants of the errors are sharp. Applications in numerical integration are also given.

Keywords: Bounded variation; Appell type polynomial; Bernoulli polynomial; Ostrowski's inequality; Trapezoidal inequality

1. Introduction

In 2000, Cerone, Dragomir and Pearce [1] proved the following trapezoid type inequalities.

Theorem 1. *Let $f : [a, b] \to R$ be a function of bounded variation. Then we have the inequality*

$$\left|\int_a^b f(t)\,\mathrm{d}t - [(x-a)f(a) + (b-x)f(b)]\right| \le \left[\frac{b-a}{2} + \left|x - \frac{a+b}{2}\right|\right] \bigvee_a^b (f) \tag{1.1}$$

for all $x \in [a, b]$, where $\bigvee_a^b (f)$ denotes the total variation of f on the interval $[a, b]$.

The inequality (1.1) is a perturbed generalization of the trapezoidal inequality for mapping of bounded variation. Using (1.1), Cerone et al. further obtained the following error estimate for the composite quadrature rule.

Theorem 2. *Let f be defined as in Theorem 1; then we have*

$$\int_a^b f(t)\,\mathrm{d}t = \sum_{i=0}^{n-1} [(\xi_i - x_i) f(x_i) + (x_{i+1} - \xi_i) f(x_{i+1})] + R(f). \tag{1.2}$$

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The remainder term $R(f)$ satisfies the estimate

$$|R(f)| \le \left[\frac{\nu(l)}{2} + \max_{i=0,1,\ldots,n-1}\left|\xi_i - \frac{x_i + x_{i+1}}{2}\right|\right] \bigvee_a^b(f) \le \nu(l) \bigvee_a^b(f), \tag{1.3}$$

where $\nu(l) := \max\{l_i | i = 0, 1, \ldots, n-1\}$, $l_i = x_{i+1} - x_i$ *and* $\xi_i \in [x_i, x_{i+1}]$.

In this paper, following the main ideas of Vinogradov [2], we give a unified treatment of error estimates for a general quadrature rule satisfying $f^{(n-1)}$ being of bounded variation. Using the perturbed inequality, we obtain the error bounds for the mid-point, trapezoid and Simpson quadrature formulas. We also generalize Euler trapezoid formulas [3].

2. The main results

A sequence of polynomials $\{u_k\}_0^\infty$ is called a sequence of Appell type polynomials if $u_0 = 1$, $u'_k = u_{k-1} (k \in Z_+)$.

Lemma 1. *Let $f : [a, b] \to R$ be such that $f^{(n-1)}$ is a function of bounded variation on $[a, b]$ for some $n \ge 1, n \in Z_+$. Moreover, if $n = 1$, $f(t)$ is continuous at x, $x \in [a, b]$. Suppose that $\{r_k\}$, $\{s_k\}$ are sequences of Appell type polynomials on $[a, x)$ and $\{u_k\}$, $\{v_k\}$ are sequences of Appell type polynomials on $(x, b]$. Let $m \in N, m \le n$,*

$$k_n(x,t) = \begin{cases} p_n(t) = r_{n-m}(t)s_m(t), & t \in [a,x); \\ q_n(t) = u_{n-m}(t)v_m(t), & t \in (x,b]. \end{cases}$$

Then we have the following equality:

$$\int_a^b f(t)\,\mathrm{d}t - \frac{(-1)^n}{C_n^m}\int_a^b k_n(x,t)\,\mathrm{d}f^{(n-1)}(t)$$
$$= \begin{cases} \dfrac{1}{C_n^m}\displaystyle\sum_{k=0}^{n-1}(-1)^{n-1-k}\Big[q_n^{(k)}(b)f^{(n-1-k)}(b) \\ \quad - q_n^{(k)}(a+)f^{(n-1-k)}(a)\Big], & x = a; \\ \dfrac{1}{C_n^m}\displaystyle\sum_{k=0}^{n-1}(-1)^{n-1-k}\Big[(p_n^{(k)}(x-) - q_n^{(k)}(x+))f^{(n-1-k)}(x) \\ \quad + q_n^{(k)}(b)f^{(n-1-k)}(b) - p_n^{(k)}(a)f^{(n-1-k)}(a)\Big], & x \in (a,b); \\ \dfrac{1}{C_n^m}\displaystyle\sum_{k=0}^{n-1}(-1)^{n-1-k}\Big[p_n^{(k)}(b-)f^{(n-1-k)}(b) \\ \quad - p_n^{(k)}(a)f^{(n-1-k)}(a)\Big], & x = b, \end{cases}$$

where $C_n^m = \frac{n!}{m!(n-m)!}$.

Proof. Integrating by parts in the sense of Riemann and Stieltjes, we can easily obtain Lemma 1. □

Remark 1. Actually, $f^{(n-1)}$ is continuous if it is of bounded variation when $n > 1$. If $k_1(x, t)$ is continuous at x, we can weaken the conditions of Lemma 1. In this case, it is not necessary that $f(t)$ is continuous at x.

Theorem 3. *Let f be defined as in Lemma 1. Suppose that $m \in N, n \in Z_+$, $m \le n$ and $\lambda \in [0, 1]$. Then we have*

$$\Bigg|\int_a^b f(t)\,\mathrm{d}t - \frac{1}{C_n^m}\sum_{j=0}^{n-1}\left[\sum_{i=L}^{U} C_j^i C_{n-j}^{n-m-i}(1-\lambda)^{m-j+i}\right]\frac{(b-x)^{n-j} - (a-x)^{n-j}}{(n-j)!}f^{(n-1-j)}(x)$$
$$- \frac{1}{C_n^m}\sum_{j=n-m}^{n-1} C_j^{n-m}\lambda^{n-j}\frac{(x-a)^{n-j}f^{(n-1-j)}(a) - (x-b)^{n-j}f^{(n-1-j)}(b)}{(n-j)!}\Bigg|$$