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Note on transport equation and fractional Sumudu transform

Abdelouahab Kadem [a], Adem Kılıçman [b,*]

[a] LMFN Mathematics Department, University of Setif, Algeria

[b] Department of Mathematics, University Putra Malaysia, 43400 UPM, Serdang, Selangor, Malaysia

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ABSTRACT

In this paper, the Chebyshev polynomials to solve analytically the fractional neutron transport equation in one-dimensional plane geometry are used. The procedure is based on the expansion of the angular flux in terms of the Chebyshev polynomials. The obtained system of fractional linear differential equation is solved analytically by using fractional Sumudu transform.

1. Introduction

In the literature, there are several integral transforms widely used in physics, astronomy as well as in engineering. In [1], Watugala introduced a new transform and named as Sumudu transform which is defined over the set of functions

$$A = \left\{ f(t) : \exists M, \tau_1, \tau_2 > 0, |f(t)| < M e^{t/\tau_i}, \text{ if } t \in (-1)^i \times [0, \infty) \right\} \tag{1.1}$$

by the following formula:

$$G(u) = S[f(t); u] =: \int_0^\infty f(ut) e^{-t} dt, \quad u \in (-\tau_1, \tau_2). \tag{1.2}$$

Since then this new transform has been applied to several problems ranging from ordinary differential equations to the problems in control engineering; see [2,3,1]. In [4], some fundamental properties of this transform were established.

In [5], the transform was extended to the distributions(generalized functions) and some of their properties were also studied in [6,7]. Recently Kılıçman et al. have applied this transform to solve the system of differential equations; see [8]. The inversion of the transformed coefficients is obtained by using *Trzaska's* method [9] and the Heaviside expansion technique.

In one of our recent work, we have presented a new approximation for solving the one dimensional transport equation by combining the Chebyshev polynomials and Sumudu transform [10]. The approach was based on the expansion of the angular flux in a truncated series of Chebyshev polynomials in the angular variable. Similarly, the present authors considered a method for the solution of the neutron transport equation in three-dimensional case by using the Walsh function, Chebyshev polynomials and the Legendre polynomials as well as the Tau method; see [11]. Similarly, the fractional transform was applied to the Maxwell equations by using the spatial function coefficients; see [12,13].

This work is devoted to the study of the fractional neutron transport equation in one-dimensional plane geometry by using the Chebyshev polynomials. The procedure is based on the expansion of the angular flux in terms of the Chebyshev

* Corresponding author.
E-mail addresses: abdelouahabk@yahoo.fr (A. Kadem), akilicman@putra.upm.edu.my (A. Kılıçman).

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polynomials. Then the resulting system of fractional linear differential equation is solved analytically by using a fractional Sumudu transform. The paper is organized as follows. Section 2 contains preliminaries, and Section 3 describes how to convert a transport equation into *FDE*; in Section 4, we report specifications and application of the method.

Let us consider the following mono-energetic 3-D transport equation:

$$\underline{\Omega}.\underline{\nabla}(\underline{r},\underline{\Omega})+\sigma_t\Psi(\underline{r},\underline{\Omega})=\int_{4\pi}\sigma_s(\underline{\Omega},\underline{\Omega}')\Psi(\underline{r},\underline{\Omega}')\mathrm{d}\Omega'+\frac{1}{4\pi}Q(\underline{r}) \tag{1.3}$$

$$\underline{\Omega}=(\eta,\xi)=\text{angular variable}, \tag{1.4}$$

and

$$\sigma_s(\mu_0)=\sum_{k=0}^{\infty}\frac{2k+1}{4\pi}\sigma_{sk}P_k(\mu_0)=\text{ differential scattering cross section}, \tag{1.5}$$

with $\mu_0=\underline{\Omega}.\underline{\Omega}'$ and $P_k=$ the kth Legendre polynomial.

2. Preliminaries

We enlist some definitions and basic results [14–16].

Definition 1. A real function $f(x),\ x>0$ is said to be in the space $C_{\alpha,\alpha\in R}$ if there exists a real number $p(>\alpha)$, such that $f(x)=x^pf_1(x)$ where $f_1(x)\in C[0,\infty)$. Clearly $C_\alpha\subset C_\beta$ if $\beta\le\alpha$.

Definition 2. A function $f(x),\ x>0$ is said to be in space $C_\alpha^m,\ m\in N\cup\{0\}$, if $f^{(m)}\in C_\alpha$.

Definition 3. The (left sided) Riemann–Liouville fractional integral of order $\mu>0$, of a function $f\in C_\mu,\ \mu\ge 1$ is defined as:

$$I^\mu f(t)=\frac{1}{\Gamma(\mu)}\int_0^t(t-\tau)^{\mu-1}f(\tau)\mathrm{d}\tau,\quad \mu>0,\ t>0, \tag{2.1}$$
$$I^0f(t)=f(t).$$

Definition 4. The (left sided) Riemann–Liouville fractional derivative of f, where $f\in C_{-1}^m$, and $m\in N\cup\{0\}$ of order $\mu>0$, is defined as:

$$D^\mu f(t)=\frac{\mathrm{d}^m}{\mathrm{d}t^m}I^{m-\mu}f(t),\quad m-1<\mu\le m,\ m\in N. \tag{2.2}$$

Since the Riemann–Liouville approach to the fractional derivative began with an expression for the repeated integration of a function, an alternative fractional derivative was introduced by Caputo in [17], and produces a derivative that has different properties: it produces zero from constant functions and, more importantly, the initial value terms of the Laplace transform are expressed by means of the values of that function and of its derivative of integer order rather than the derivatives of fractional order as in the Riemann–Liouville derivative.

Definition 5. The (left sided) Caputo fractional derivative of f, $f\in C_{-1}^m$, and $m\in N\cup\{0\}$ of order $\mu>0$, is defined as:

$$D_c^\mu f(t)=\begin{cases}\left[I^{m-\mu}f^{(m)}(t)\right] & m-1<\mu\le m,\ m\in N,\\ \dfrac{\mathrm{d}^m}{\mathrm{d}t^m}f(t) & \mu=m.\end{cases} \tag{2.3}$$

Note that the relation between the Riemann–Liouville and Caputo fractional derivatives can be given as follows.

(i) $I^\mu t^\gamma=\frac{\Gamma(\gamma+1)}{\Gamma(\gamma+\mu+1)}t^{\gamma+\mu},\ \mu>0,\gamma>-1,\ t>0.$

(ii) $I^\mu D_c^\mu f(t)=f(t)-\sum_{k=0}^{m-1}f^{(k)}(0_+)\frac{t^k}{k!},\ m-1<\mu\le m,\ m\in N^*.$

(iii) $D_c^\mu f(t)=D^\mu\left(f(t)-\sum_{k=0}^{m-1}f^{(k)}(0_+)\frac{t^k}{k!}\right),\ m-1<\mu\le m,\ m\in N^*.$

(iv) $D^\beta I^\mu f(t)=\begin{cases}I^{\mu-\beta}f(t) & \text{if }\mu>\beta,\\ f(t) & \text{if }\mu=\beta,\\ D^{\beta-\mu}f(t) & \text{if }\mu<\beta,\end{cases}$

(v) $D_c^\mu D^m f(t)=D^{\mu+m}f(t),\ m=0,1,2,\ldots,\ m-1<\mu<m,$

see [18].

From here on, we will use $\mathcal{C}_\gamma([a,b])(\gamma\in R)$ to denote the Banach space

$$\mathcal{C}_\gamma([a,b])=\left\{g(x)\in C((a,b]):\|g\|_{\mathcal{C}_\gamma}=\|(x-a)^\gamma g(x)\|_C<\infty\right\}. \tag{2.4}$$

In particular, $\mathcal{C}_0([a,b])$ represents the space of continuous functions in $[a,b]$, that is, $C([a,b])$.