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On (ψ, ϕ)-weakly contractive condition in partially ordered metric spaces

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Recently, Heman Kumar Nashine and Bessem Samet [H.K. Nashine, B. Samet, Fixed point results for mappings satisfying (ψ, ϕ)-weakly contractive condition in partially ordered metric spaces, Nonlinear Anal. 74 (2011) 2201–2209] studied some coincidence fixed point and common fixed point theorems for two mappings satisfying (ψ, ϕ)-weakly contractive condition in an ordered complete metric space. In the present paper, we study some coincidence fixed point and common fixed point theorems for three mappings S, T and R satisfying (ψ, ϕ)-weakly contractive condition in an ordered complete metric space, where the mappings S and T are assumed to be weakly increasing with respect to R. Our results generalize several well-known results in the literature.

1. Introduction

The Banach contraction mapping principle [1] is a very popular tool for solving existence problems in many branches in mathematical analysis. Generalization of the Banach principle has been a heavily investigated branch of research; see [2–12]. In particular, there has been a number of works involving altering distance functions. There are control functions which alter the distance between two points in a metric space. Such functions were introduced by Khan et al. [13].

Definition 1.1 (*Altering Distance Function, [13]*)**.** A function $\phi : [0, +\infty) \to [0, +\infty)$ is called an altering distance function if the following properties are satisfied:

(i) ϕ is continuous and nondecreasing,
(ii) $\phi(t) = 0$ if and only if $t = 0$.

Khan et al. [13] proved the following theorem.

Theorem 1.1. *Let (X, d) be a complete metric space, ψ an altering distance function and $T : X \to X$ satisfying*

$$\psi(d(Tx, Ty)) \le c\psi(d(x, y))$$

for all $x, y \in X$, where $0 < c < 1$. Then T has a unique fixed point.

Alber and Guerre-Delabriere [14] introduced the definition of weak ϕ-contraction.

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Definition 1.2 (See [14]). A self mapping T on a metric space X is called weak ϕ-contraction if there exists a function $\phi : [0, +\infty) \to [0, +\infty)$ such that

$$d(Tx, Ty) \le d(x, y) - \phi(d(x, y))$$

for all $x, y \in X$.

The notion of ϕ-contraction and weak ϕ-contraction has been studied by many authors [15,16,4–6,10,17,12,18]. In [12], the following theorem was proved.

Theorem 1.2 (See [12]). *Let (X, d) be a complete metric space. Let $T : X \to X$ be a mapping such that for all $x, y \in X$,*

$$d(Tx, Ty) \le d(x, y) - \phi(d(x, y)),$$

where ϕ is an altering distance function. Then T has a unique fixed point.

While Dutta and Choudhury [4] proved the following result.

Theorem 1.3 (See [4]). *Let (X, d) be a complete metric space and $T : X \to X$ be a mapping such that for all $x, y \in X$,*

$$\psi(d(Tx, Ty)) \le \psi(d(x, y)) - \phi(d(x, y)),$$

where ψ and ϕ are both altering distance functions. Then T has a unique fixed point.

In recent years, many results appeared related to fixed point theorems in a complete ordered metric space. Ran and Reurings [11] extended the Banach contraction principle in partially ordered sets with some applications to linear and nonlinear matrix equations. While Nieto and Rodŕiguez-López [7] extended the result of Ran and Reurings and applied their main theorems to obtain a unique solution for a first order ordinary differential equation with periodic boundary conditions. Bhaskar and Lakshmikantham [19] introduced the concept of a mixed monotone mappings and obtained some coupled fixed point results. Also, they applied their results on a first order differential equation with periodic boundary conditions. Further improvements of Bhaskar and Lakshmikantham results were found independently as example in [20–30].

Harjani and Sadarangani [5,6] obtained some fixed point theorems in a complete ordered metric space using altering distance functions. They proved the following theorems.

Theorem 1.4 (See [6]). *Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a metric d in X such that (X, d) is a complete metric space. Let $f : X \to X$ be a continuous and nondecreasing mapping such that*

$$\psi(d(fx, fy)) \le \psi(d(x, y)) - \phi(d(x, y))$$

for comparable $x, y \in X$, where ψ and ϕ are altering distance functions. If there exists $x_0 \preceq f(x_0)$, then f has a fixed point.

Theorem 1.5 (See [6]). *Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a metric d in X such that (X, d) is a complete metric space. Assume that X satisfies if (x_n) is a nondecreasing sequence in X such that $x_n \to x$, then $x_n \preceq x$ for all $n \in \mathbb{N}$. Let $f : X \to X$ be a nondecreasing mapping such that*

$$\psi(d(fx, fy)) \le \psi(d(x, y)) - \phi(d(x, y))$$

for comparable $x, y \in X$, where ψ and ϕ are altering distance functions. If there exists $x_0 \preceq f(x_0)$, then f has a fixed point.

Recently, Nashine and Samet proved the following results.

Theorem 1.6 (See [10]). *Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a metric d on X such that (X, d) is a complete metric space. Let $T, R : X \to X$ be given mappings such that for all $x, y \in X$ with Rx and Ry are comparable, we have*

$$\psi(d(Tx, Ty)) \le \psi(d(Rx, Ry)) - \phi(d(Rx, Ry)),$$

where ϕ and ψ are altering distance functions. Assume that T and R satisfy the following hypotheses:

(i) *T is weakly increasing with respect to R,*
(ii) *$TX \subseteq RX$,*
(iii) *T and R are continuous,*
(iv) *the pair $\{T, R\}$ is compatible.*

Then T and R have a coincidence point, that is, there exists $u \in X$ such that $Ru = Tu$.

Theorem 1.7 (See [10]). *Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a metric d on X such that (X, d) is a complete metric space. Let $T, R : X \to X$ be mappings such that for all $x, y \in X$ with Rx and Ry are comparable, we have*

$$\psi(d(Tx, Ty)) \le \psi(d(Rx, Ry)) - \phi(d(Rx, Ry)), \tag{1}$$

where ϕ and ψ are altering distance functions. Suppose the following hypotheses: