Computers and Mathematics with Applications 66 (2013) 62–80

Contents lists available at SciVerse ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

A way of updating the density function for the design of the drum

Xiangyin Kong, Zhengda Huang *

Department of Mathematics, Zhejiang University, Hangzhou, 310027, PR China

ARTICLE INFO

Article history:
Received 21 June 2012
Received in revised form 31 January 2013
Accepted 6 April 2013

Keywords:
Finite element method
Gâteaux derivative
Eigenvalue
Shape optimization
The weight

ABSTRACT

Designing an acoustic drum can be categorized into a class of eigenvalue optimization problems in the structural engineering area. In this paper, we propose an algorithm that is based on the Gâteaux derivative of the objective function with respect to the density functions and analyze our algorithm in detail. In the algorithm, we deal with the geometry constraint by exchanging the densities of two domains occupied by two kinds of different materials. Finally we apply this algorithm to some practical examples frequently used by the researchers and present some numerical results to show its feasibility, stability and efficiency.

1. Introduction

In this paper, we mainly solve a class of problems on designing an acoustic drum in the structure engineering design area. This class of problems can be modeled by the equation

$$\begin{cases} -\Delta u = \lambda \rho(x) u, & x \in \Omega \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{1}$$

where $\Omega \subseteq R^2$ (or R^3), denoting the head of a drum, is a convex, bounded and connected closed domain, and the function $\rho(x) : \Omega \to R$ denotes the density of the drum. In general, the drum is made up of m different kinds of materials with the i-th kind of material's density c_i $(i = 1, 2, \ldots, m)$ and $0 < c_1 < c_2 < \cdots < c_m$.

This class of problems generally involves some geometrical constraints or other ones. If we let $\Omega_i = \{x \in \Omega | \rho(x) = c_i\}$ $(i = 1, 2, \ldots, m)$, then the geometrical constraints can be described as $\sum_{i=1}^{m} K_i = |\Omega|$ with $|\Omega_i| = K_i > 0$, where $|\cdot|$ is the area of a domain and K_i is a preassigned number, $i = 1, 2, \ldots, m$. For the sake of convenient description, we denote by $\mathcal{A}_d(\Omega)$ the set of density functions ρ that satisfy the geometrical constraints. This model can also be used to determine the shape of vibrating membranes which is comprised of different materials [1–3] or to devise the lightest structure with certain compliance constraint [4], and can be solved by finding the optimal function $\rho(x) \in \mathcal{A}_{\mathrm{d}}(\Omega)$.

When $m = 2$, it is derived by Cox [5,6] that the weak form of Eq. (1)

$$\int_\Omega \nabla u \cdot \nabla v dx = \int_\Omega \lambda \rho u v dx, \quad \forall v \in H_0^1(\Omega) \tag{2}$$

has a sequence of nontrivial eigenvalues

$$0 < \lambda_1(\Omega, \rho) < \lambda_2(\Omega, \rho) \leq \cdots \to \infty, \tag{3}$$

where that the eigenvalue λ_1 is simple, or, that the first two eigenvalues λ_1 and λ_2 are distinct is proved in Lemma 3.6 of [5].

* Corresponding author.
E-mail addresses: consosion@163.com (X. Kong), zdhuang@zju.edu.cn (Z. Huang).

0898-1221/$ – see front matter
http://dx.doi.org/10.1016/j.camwa.2013.04.012

To let our solution make sense, we assume that λ_1 is simple and take the assumption in [7] that λ_2 is separated from λ_3 for Ω with any composition. When we say the density function $\rho_{\text{opt}} \in A_{\text{d}}(\Omega)$ is optimal, we mean that it is the corresponding solution to the following three problems

$$\begin{aligned}
&\text{(I) } \lambda_1(\rho_{\text{opt}}) = \max_{\rho \in A_{\text{d}}(\Omega)} \lambda_1(\rho);\\
&\text{(II) } \lambda_1(\rho_{\text{opt}}) = \min_{\rho \in A_{\text{d}}(\Omega)} \lambda_1(\rho);\\
&\text{(III) } \lambda_2(\rho_{\text{opt}}) - \lambda_1(\rho_{\text{opt}}) = \max_{\rho \in A_{\text{d}}(\Omega)} \{\lambda_2(\rho) - \lambda_1(\rho)\}.
\end{aligned} \tag{4}$$

This class of problems is hard to solve because of the lack of the topology information of the optimal shape, and there must be a good mechanism to express the composition of materials. The mostly used techniques now are the homogenization method [8,9], the level set method [10] and so on.

For $m = 2$, algorithms in the existing literature for this class of problems can be categorized into two classes. Algorithms of the first class are to transform the original constrained optimization problem into an unconstrained one by means of the Lagrangian multiplier method or the augmented Lagrangian method. Osher and Santosa [7] employed the conventional level set method proposed in [10], the variational level set calculus presented in [11], and the projected gradient method used in [12], to construct an efficient algorithm for three aforementioned problems in 2D. Haber [13] combined the reduced Hessian sequential quadratic programming method with multilevel continuation technique to put forward a rapid and robust algorithm to maximize or minimize λ_1 in 2D and 3D, and approximated the eigenvalue equation with the inverse iteration method instead of solving a generalized matrix eigenvalue problem in each iteration. Strang and Persson [14] took the finite element method to solve the first two problems in (4) when the domain Ω is irregular in 2D.

As pointed out in [15–19], the level set method which only involves the shape sensitivity may get stuck at shapes with fewer holes than the optimal geometry in some applications such as the structure designs. How to avoid the occurrence of such phenomenon has attracted many researchers' attentions. He et al. [18] solved a class of shape optimization problems in 2D by incorporating the topological derivative into the shape derivative based level set methods. Zhu et al. [19] made use of the piecewise constant level set method proposed in [20], the Lagrange multiplier method and the augmented Lagrange methods to propose three efficient iteration algorithms for three above-mentioned problems and some other optimization problems. Zhu et al. [21] also used the binary level set method in [22], the Lagrange multiplier method and the augmented Lagrange methods to propose two effective and robust algorithms.

The second class of algorithms, which are to deal with the geometrical constraints while updating the density function ρ, are less common in the literature. Krein [23] considered the least eigenvalue problem when the domain Ω is a disk, he filled the inner part of the domain with high density material and the remaining parts with low density material. For a high dimensional case, Cox first proved in [6] that the interface between two kinds of materials lies in a level set of the eigenfunction of λ_1, then proved in [24] that there exists a constant C_1 such that

$$\rho(x) = \begin{cases} \rho_1, & u_1 \le C_1, \\ \rho_2, & u_1 > C_1, \end{cases} \tag{5}$$

where (λ_1, u_1) is the first eigenpair to Eq. (1), and finally gave an algorithm in [25] to seek the constant C_1. Zhang et al. [26] solved the least eigenvalue problem for the multi-domain and multi-material cases ($m \ge 2$) by minimizing the quantity

$$R(\Omega, \rho) = \frac{1}{\int_\Omega \rho u^2 dx}. \tag{6}$$

Zhang et al. [27] proposed two types of greedy algorithms which are based on the expression of λ for three above-mentioned problems. Based on the Rayleigh quotient formulation of eigenvalues, Kao and Su [28] proposed an efficient rearrangement algorithm to solve minimization and maximization of the k-th eigenvalue ($k \ge 2$) and maximization of spectrum ratios of the second order elliptic differential operator in 2D.

In the paper, for problem (II) in (4), since the eigenvalues of Eq. (1) are positive, we can obtain the minimal value of λ_1 by first maximizing $-\lambda_1$ and then taking the negative of the maximum of $-\lambda_1$. In such way, these three problems in (4) can all be solved by maximizing the objective function, and the objective functions are λ_1, $-\lambda_1$ and $\lambda_2 - \lambda_1$, respectively. Inspired by the way that Ω is divided into equal-area rectangles in [26], we devise a new algorithm which can be used for three problems by introducing a different scheme of updating the density function in virtue of the Gâteaux derivative of the objective functions with respect to the density function and the equal-area subdivision. This updating scheme tries to keep the increment of the objective function positive before and after updating the density function and make the updated density function ρ satisfy the geometrical constraint at the same time. We also discuss the effects of the number of triangles whose densities are changed to the variation of the density function and choose an appropriate value of this number by trial-and-error method. The algorithm can be categorized into the second class and can get satisfactory approximations in relatively fewer steps, compared with some algorithms in [19,21,27] for some frequently used numerical examples. When $m = 2$, our algorithm is similar to the algorithm in [28], and the differences between these two methods will be discussed in Remark 3 of Section 2.