Available online at www.sciencedirect.com
ScienceDirect
Computers and Mathematics with Applications 53 (2007) 413–435
An International Journal
computers & mathematics with applications
www.elsevier.com/locate/camwa

Solvability of the problem of the self-propelled motion of several rigid bodies in a viscous incompressible fluid

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Received 13 September 2005; accepted 27 February 2006

Abstract

In this paper, we investigate the problem of the motion of self-propelled rigid bodies in a viscous incompressible fluid filling a bounded container. The motion of the fluid is governed by the Navier–Stokes equations. The bodies move due both to the flow of the ambient fluid and to the engines which are modelled by fluxes of the fluid through the boundaries of the bodies. It is proved that the problem has at least one weak solution on an arbitrary time interval which does not include instants of collisions of the bodies.

Keywords: Navier–Stokes equations; Rigid body; Self-propelled motion

1. Introduction

The subject of this paper is the problem of the motion of self-propelled rigid bodies immersed in a viscous incompressible fluid. We suppose that the fluid flow is governed by the Navier–Stokes system, whereas the bodies move according to the classical balance equations for linear and angular momentum. The motion of the bodies is caused by the external bulk forces (e.g., gravity), by the stresses in the surrounding fluid, and by the propulsive forces which occur due to additional fluid fluxes produced by the bodies on their boundaries. Since the fluid flow, in its turn, is influenced by the bodies, we have a coupled mechanical system whose evolution is described by a system of nonlinear partial and ordinary differential equations. Let us describe the classical statement of the problem.

Suppose that N rigid bodies move in a bounded domain $\Omega \subset \mathbb{R}^3$. We denote by $\mathcal{S}_t^k$ ($k = 1, \ldots, N$) the subdomain of Ω occupied by the k-th body at the time instant t. The boundary of $\mathcal{S}_t^k$ will be denoted by $\mathcal{G}_t^k$. The fluid fills the domain $\mathcal{F}_t = \Omega \setminus (\mathcal{S}_t \cup \mathcal{G}_t)$, where $\mathcal{S}_t = \cup_{k=1}^N \mathcal{S}_t^k$ and $\mathcal{G}_t = \cup_{k=1}^N \mathcal{G}_t^k$. Sometimes, it will be convenient to consider the exterior of Ω as an immovable rigid body and to denote $\mathcal{S}_t^0 = \mathbb{R}^3 \setminus \overline{\Omega}$, $\mathcal{G}_t^0 = \partial\Omega$.

Let $\delta_{ij}(t)$ be the distance between the sets $\mathcal{S}_t^i$ and $\mathcal{S}_t^j$ with $i, j \in \{0, 1, \ldots, N\}$. In this paper, the collisions of the bodies will not be admitted and we will solve the problem on a time interval $[0, T]$, $T < \infty$, such that

$$\delta(t) = \min_{i,j=0,\ldots,N} \delta_{ij}(t) \geqslant \delta_* > 0 \quad \text{for all } t \in [0, T]. \tag{1.1}$$

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0898-1221/$ - see front matter
doi:10.1016/j.camwa.2006.02.045

Here, δ_* is an arbitrary positive number.

Denote by $\boldsymbol{v}$ the velocity of the fluid. According to our assumption, the vector field $\boldsymbol{v}$ is defined in $\mathcal{F}_t$ and satisfies the Navier–Stokes equations:

$$\rho_F\,(\boldsymbol{v}_t + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v}) = \operatorname{div} P(\boldsymbol{v}) + \rho_F \boldsymbol{f}, \tag{1.2}$$

$$\operatorname{div}\boldsymbol{v} = 0, \tag{1.3}$$

$$P(\boldsymbol{v}) = -pI + 2\mu D(\boldsymbol{v}), \tag{1.4}$$

and the boundary condition:

$$\boldsymbol{v}(\boldsymbol{x},t) = \boldsymbol{g}_0(\boldsymbol{x},t) \quad \text{for } \boldsymbol{x}\in\partial\Omega, \tag{1.5}$$

where ρ_F is the density of the fluid, $\boldsymbol{f}$ the external bulk force, p the pressure, μ the viscosity, I the identity tensor, and $D(\boldsymbol{v})$ the deformation rate tensor which has the components

$$D_{ij}(\boldsymbol{v}) = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right).$$

We assume that ρ_F and μ are positive constants. Notice that (1.5) is understood in the sense of traces. To simplify the notations, we will omit the symbol of the trace operator. The function $\boldsymbol{g}_0$ is defined on $\mathbb{R}^3\times[0,T]$ and its trace represents the flux of the fluid through $\partial\Omega$. We suppose that $\operatorname{div}\boldsymbol{g}_0 = 0$.

Let us denote by $\boldsymbol{u}$ the velocity field in the set $\mathcal{S}_t$ occupied by the bodies. Since the bodies are assumed to be rigid, there exist vectors $\boldsymbol{a}_k(t)$ and skew-symmetric matrices $Q_k(t)$, $k = 1,\dots,N$, such that

$$\boldsymbol{u}(\boldsymbol{x},t) = \boldsymbol{a}_k(t) + Q_k(t)\left(\boldsymbol{x} - \hat{\boldsymbol{x}}_k(t)\right) \quad \text{for } \boldsymbol{x}\in\mathcal{S}_t^k, \tag{1.6}$$

where

$$\hat{\boldsymbol{x}}_k(t) = |\mathcal{S}_t^k|^{-1}\int_{\mathcal{S}_t^k}\boldsymbol{x}\,\mathrm{d}\boldsymbol{x}$$

is the mass center of the k-th body (the bodies are assumed to be homogeneous) and $|\mathcal{A}|$ stands for the three-dimensional Lebesgue measure of a set $\mathcal{A}$. Notice that there exist vectors $\boldsymbol{\omega}_k(t)$, $k = 1,\dots,N$, such that

$$Q_k(t)\boldsymbol{\xi} = \boldsymbol{\omega}_k(t)\times\boldsymbol{\xi} \quad \text{for all } \boldsymbol{\xi}\in\mathbb{R}^3, \tag{1.7}$$

where the sign "$\times$" stands for the vector product. The vector $\boldsymbol{a}_k$ represents the velocity of the mass center of the k-th body, i.e., $\boldsymbol{a}_k(t) = \mathrm{d}\hat{\boldsymbol{x}}_k(t)/\mathrm{d}t$, and $\boldsymbol{\omega}_k(t)$ is its angular velocity.

According to the laws of classical mechanics, the functions $\boldsymbol{a}_k(t)$ and $\boldsymbol{\omega}_k(t)$, $k = 1,\dots,N$, satisfy the following equations:

$$m_k\frac{\mathrm{d}\boldsymbol{a}_k}{\mathrm{d}t} = \int_{\mathcal{G}_t^k} P(\boldsymbol{v})\boldsymbol{n}\,\mathrm{d}s + \int_{\mathcal{S}_t^k}\rho_S^k\boldsymbol{f}\,\mathrm{d}\boldsymbol{x} + \int_{\mathcal{G}_t^k}\rho_F\boldsymbol{v}(\boldsymbol{u}-\boldsymbol{v})\cdot\boldsymbol{n}\,\mathrm{d}s, \tag{1.8}$$

$$\rho_S^k\frac{\mathrm{d}J_k\boldsymbol{\omega}_k}{\mathrm{d}t} = \int_{\mathcal{G}_t^k}(\boldsymbol{x}-\hat{\boldsymbol{x}}_k)\times P(\boldsymbol{v})\boldsymbol{n}\,\mathrm{d}s + \int_{\mathcal{S}_t^k}\rho_S^k(\boldsymbol{x}-\hat{\boldsymbol{x}}_k)\times\boldsymbol{f}\,\mathrm{d}\boldsymbol{x} + \int_{\mathcal{G}_t^k}\rho_F(\boldsymbol{x}-\hat{\boldsymbol{x}}_k)\times\boldsymbol{v}(\boldsymbol{u}-\boldsymbol{v})\cdot\boldsymbol{n}\,\mathrm{d}s, \tag{1.9}$$

where m_k is the mass of the k-th body, ρ_S^k the density, $\boldsymbol{n}$ the normal to $\mathcal{G}_t$ directed towards the fluid, and $J_k = \int_{\mathcal{S}_t^k}\left(|\boldsymbol{x}-\hat{\boldsymbol{x}}_k|^2 I - (\boldsymbol{x}-\hat{\boldsymbol{x}}_k)\otimes(\boldsymbol{x}-\hat{\boldsymbol{x}}_k)\right)\mathrm{d}\boldsymbol{x}$ the matrix of the inertia moments of the k-th body related to its mass center. We will suppose that ρ_S^k, $k = 1,\dots,N$, are positive constants.

The right-hand side of (1.8) is the sum of three forces acting on the k-th body. The first one is the fluid stress on the boundary of the body. The second one is the external bulk force. The third term represents the propulsive force that occurs due to the flux of the fluid through the boundary of the body. We assume that this flux is prescribed and, since it is in fact the difference between the traces of the velocity fields $\boldsymbol{v}$ and $\boldsymbol{u}$ on $\mathcal{G}_t$, we have the following condition:

$$\boldsymbol{v}(\boldsymbol{x},t) - \boldsymbol{u}(\boldsymbol{x},t) = \boldsymbol{w}(\boldsymbol{x},t) \quad \text{for } \boldsymbol{x}\in\mathcal{G}_t, \tag{1.10}$$

where the function $\boldsymbol{w}$ will be specified later (see Section 2.2).