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A remark on characterization of entropy solutions

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Abstract

Colombeau's algebra of generalized functions is used to study the solutions to a single hyperbolic conservation law. In a simple setting of travelling shocks, we formulate a new interesting necessary and sufficient condition for the solution to be entropic.

Keywords: Hyperbolic conservation law; Entropy solution; Travelling shocks; Colombeau's generalized functions

1. Introduction

Since their invention, partial differential equations became one of the most useful tools in describing the physical reality. Yet one has also to admit that despite numerous successful applications, there is, from the purely mathematical point of view, certain uneasiness connected with the following fact: on the one hand, most PDEs typically involve derivatives of the 1st and 2nd order; on the other hand, one is often not able to prove the very existence of solutions smooth enough so that these derivatives can be taken. Even worse, one sometimes knows that it is also physically unreasonable to expect the smooth solutions to exist. This is typically the case with the equations where the nonlinearities are present.

Thus, in order to give the meaning to our equations, a generalized concept of derivative is needed. Here the space of distributions $\mathscr{D}'$, introduced by L. Schwartz around 1950, is commonly regarded as the most general framework for the theory of PDEs. However, there is also a well-known drawback connected to the space $\mathscr{D}'$, which consists in the following: working in $\mathscr{D}'$, one surely has an unlimited access to all linear operations (including taking derivatives, Fourier transform, change of variables). On the other hand, there is no general approach as long as the nonlinear operations are concerned. Even the pairwise multiplication of elements of $\mathscr{D}'$ is generally not possible, as already known since the times of Schwartz, [1]; see also [2] for a more detailed discussion.

Another, maybe a more subtle weakness of the space $\mathscr{D}'$ becomes apparent when studying the equation

$$u_t + [F(u)]_x = 0.$$

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This is clearly one of the simplest nonlinear PDEs. It can be shown by means of simple examples that global smooth solutions cannot exist — on the other hand, when one starts to work in $\mathscr{D}'$, the solutions are no longer unique. See e.g. [3].

In order to ensure uniqueness, one introduces the so-called entropy solutions. These are motivated by the following formal argument: we carry out the differentiation by x and multiply by $\eta'(u)$, assuming that $F'\eta' = \psi'$. Thus

$$u_t + F'(u)u_x = 0 \quad / \cdot \eta'(u)$$
$$\eta'(u)u_t + F'(u)\eta'(u)u_x = 0$$
$$[\eta(u)]_t + [\psi(u)]_x = 0.$$

None of these steps can really be justified in $\mathscr{D}'$ if η is a nonlinear function. But if the nonlinearity of η is "one-sided", i.e., η is convex, then at least the following inequality, the so-called entropy condition

$$[\eta(u)]_t + [\psi(u)]_x \leq 0$$

is required to hold. As is well-known, see e.g. [3], the entropy condition can be justified using the vanishing viscosity method. However, the space $\mathscr{D}'$ itself would not allow us to describe what happens here.

A question naturally arises whether a more general space of functions could be devised where at least some of the aforementioned issues could be clarified. One such construction was introduced by Colombeau, see [4] or [2]. The Colombeau's space $\mathscr{G}$ generalizes both the concept of a conventional function and the distribution in a natural way. Moreover, $\mathscr{G}$ is in fact a differential algebra, that is to say, arbitrary differentiation and multiplication as well as composition with smooth functions is possible in $\mathscr{G}$, whereas the chain rule and the Leibniz rule remain valid as expected. The only limitation lies in the fact that the elements of $\mathscr{G}$ cannot be composed with functions that are less regular than C^∞. This still leaves a lot of space to applications as many naturally arising PDE have in fact analytic nonlinearities.

The distinguishing feature between $\mathscr{G}$ and $\mathscr{D}'$ lies in the fact that $\mathscr{G}$ is a sort of more subtle way of looking at things. That is, some expressions that are identical in the classical sense, say in $\mathscr{D}'$ or L^∞, need not be the same when computed in $\mathscr{G}$. The simplest examples are provided by the Heaviside function H and the Dirac distribution δ. Surprisingly, one has $H \neq H^2$ and $x\delta(x) \neq 0$ in $\mathscr{G}$, in contrast to what holds in L^∞ or $\mathscr{D}'$. See next section or [2] for details.

Nonetheless, it is precisely this ability of $\mathscr{G}$ to "see" such distinctions that we are employing in this paper to characterize the entropy solutions.

The paper is organized in the following way. In the second part, we outline the elements of Colombeau's theory that will be needed for our purpose. In the third part we prove our main result. The last part presents some examples, and discusses the relations to other characterizations of entropy solutions, as well as possible generalizations.

2. Colombeau's theory

In Colombeau's algebra of generalized functions $\mathscr{G}(\mathbb{R}^n)$ a function f from $\mathbb{R}^n$ to $\mathbb{R}$ is represented by the mapping

$$f = f(\phi, x) : \Phi \times \mathbb{R}^n \to \mathbb{R},$$

where

$$\Phi = \left\{ \phi \in \mathscr{D}(\mathbb{R}^n) : \int_{\mathbb{R}^n} \phi = 1 \right\}.$$

Here $\mathscr{D}(\mathbb{R}^n)$ is the space of C^∞ functions with bounded support. It is required that $f(\phi, \cdot)$ is for a fixed ϕ a C^∞ function. The convenient way of looking at $\mathscr{G}(\mathbb{R}^n)$ is that its elements are simply sequences of smooth functions, indexed by $\phi \in \Phi$. Certain growth conditions with respect to ϕ are also required, but we will not need that here. See [4] or [2] for details.

The underlying idea is of course the convolution. If $f : \mathbb{R}^n \to \mathbb{R}$ is a "conventional" function, say $f \in L^1_{\text{loc}}$, then the canonical embedding $i(f)$ into $\mathscr{G}$ is given by

$$i(f) = [i(f)](\phi, x) = \int_{\mathbb{R}^n} f(x + y)\phi(y)\mathrm{d}y. \tag{1}$$