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On flows of an incompressible fluid with a discontinuous power-law-like rheology

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Abstract

We establish the mathematical theory for steady and unsteady flows of fluids with discontinuous constitutive equations. We consider a model for a fluid that at certain critical values of the shear rate exhibits jumps in the generalized viscosity of a power-law type. Using tools such as Young measures, maximal monotone operators, compact embeddings and energy equality, we prove the existence of a solution to the problem under consideration. In this approach, Galerkin approximations converge strongly to the solution of the original problem.

Keywords: Maximal monotone operators; Young measures; Energy equality; Non-Newtonian fluids

1. Introduction

The list of non-Newtonian[1] phenomena exhibited by incompressible liquids typically includes (see for example [1] for their description): (i) shear thinning/shear thickening and/or pressure thickening (these are responses when the generalized viscosity decreases/increases with increasing shear rate and/or increases with increasing pressure); (ii) the presence of normal stress differences in a simple shear flow (the response closely connected with effects such as rod-climbing, die swell, etc.), (iii) viscoelastic responses such as stress relaxation and non-linear creep, and (iv) the presence of yield stress. We focus mainly on the last of these responses, which can be described as follows:

$$\begin{aligned} &\text{if } |\mathbf{T}| \le \tau^* \quad \text{then } \mathbf{D}(\boldsymbol{v}) = \mathbf{0}, \\ &\text{if } |\mathbf{T}| > \tau^* \quad \text{then } \mathbf{D}(\boldsymbol{v}) \neq \mathbf{0}, \quad \text{and} \quad \text{then } \mathbf{T} = \mathbf{f}(\mathbf{D}(\boldsymbol{v})). \end{aligned} \tag{1.1}$$

Here, $\boldsymbol{v}$ is the velocity, $\mathbf{D}(\boldsymbol{v})$ the symmetric part of the velocity gradient $\nabla \boldsymbol{v}$, $\mathbf{T}$ denotes the Cauchy stress, τ^* is the threshold value for the magnitude of $\mathbf{T}$, and $\mathbf{f}$ stands for any constitutive equation. Note that we can alternatively

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[1] A fluid is said to be non-Newtonian if its behaviour cannot be captured by the Navier–Stokes equations.

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rewrite (1.1) as

$$\begin{aligned} &\text{if } \mathbf{D}(\boldsymbol{v}) = \mathbf{0} \quad \text{then } |\mathbf{T}| \le \tau^*, \\ &\text{if } \mathbf{D}(\boldsymbol{v}) \neq \mathbf{0} \quad \text{then } |\mathbf{T}| > \tau^*, \quad \text{and} \quad \text{then } \mathbf{T} = \mathbf{f}(\mathbf{D}(\boldsymbol{v})). \end{aligned} \tag{1.2}$$

The presence of yield stress is a controversial phenomenon, since it contradicts the standard understanding of what is meant by a fluid, which is a material that cannot sustain shear stress. Thus a fluid, by its definition, is such a material that starts to flow immediately after any shear stress is applied, while (1.2) requires that Cauchy stress overshoots the critical value before the flow starts. We can, however, argue that for small magnitudes of the stress, no flow is visible within the time scale of normal observation,[2] consequently, we can view the model with the yield stress, which is also an example of a model with discontinuous Cauchy stress, as a possible and reasonable approximation of more realistic fluid response. We refer to Málek and Rajagopal [1] for a discussion of these issues.

In this article, we deal with the following "generalization" of the constitutive equation (1.2). For a given $d^* > 0$, we have

$$\begin{aligned} &\text{if } |\mathbf{D}(\boldsymbol{v})| < d^* \quad \text{then } \mathbf{T} = \mathbf{T}_1(\mathbf{D}(\boldsymbol{v})) = \nu_1(|\mathbf{D}(\boldsymbol{v})|^2)\mathbf{D}(\boldsymbol{v}), \\ &\text{if } |\mathbf{D}(\boldsymbol{v})| > d^* \quad \text{then } \mathbf{T} = \mathbf{T}_2(\mathbf{D}(\boldsymbol{v})) = \nu_2(|\mathbf{D}(\boldsymbol{v})|^2)\mathbf{D}(\boldsymbol{v}), \\ &\text{if } |\mathbf{D}(\boldsymbol{v})| = d^* \quad \text{then } \mathbf{T} = \nu^*\mathbf{D}(\boldsymbol{v}), \end{aligned} \tag{1.3}$$

where $\nu^* \in [\min\{\nu_1^-, \nu_2^+\}, \max\{\nu_1^-, \nu_2^+\}]$ with $\nu_1^- := \lim_{|\boldsymbol{\xi}| \to d^*-} \nu_1(|\boldsymbol{\xi}|^2)$ and $\nu_2^+ := \lim_{|\boldsymbol{\xi}| \to d^*+} \nu_2(|\boldsymbol{\xi}|^2)$.

We justify the model (1.3) using arguments similar to those for the yield stress phenomenon. Once the shear rate reaches a certain critical value d^*, this critical shear rate initiates series of chemical reactions that, within a very short time interval, changes the viscosity of the material dramatically. Since this change is significant and also very quick, it seems acceptable to capture this feature by a constitutive equation of the form (1.3). Note that if ν_i in (1.3) is of the form

$$\nu_i(|\boldsymbol{\xi}|^2) = \nu_{oi}|\boldsymbol{\xi}|^{r_i - 2}, \quad (i = 1, 2)$$

where $\nu_{oi} > 0$ and $r_i \in (1, \infty)$ are the model characteristics; we talk about power-law fluid response, and (1.3) then describes the change of one power-law response to another. In this paper, we consider $\mathbf{T}_1, \mathbf{T}_2$ from (1.3) so that they generalize the power-law constitutive equations in the following sense. We assume that there are fixed parameters $r, q \in (1, \infty)$, positive constants c_1, c_2, c_4, c_5 and arbitrary constants c_3, c_6 such that for all $\boldsymbol{\xi} \in \mathbb{R}^{d^2}$, we have

$$\begin{aligned} |\mathbf{T}_1(\boldsymbol{\xi})| &\le c_1(1 + |\boldsymbol{\xi}|)^{r-1}, \\ |\mathbf{T}_2(\boldsymbol{\xi})| &\le c_4(1 + |\boldsymbol{\xi}|)^{q-1}, \end{aligned} \quad \text{and} \quad \begin{aligned} \mathbf{T}_1(\boldsymbol{\xi}) \cdot \boldsymbol{\xi} &\ge c_2|\boldsymbol{\xi}|^r - c_3, \\ \mathbf{T}_2(\boldsymbol{\xi}) \cdot \boldsymbol{\xi} &\ge c_5|\boldsymbol{\xi}|^q - c_6. \end{aligned} \tag{1.4}$$

In addition, we assume that $\mathbf{T}_1, \mathbf{T}_2$ are strictly monotone, i.e., for $i = 1, 2$ we have

$$(\mathbf{T}_i(\boldsymbol{\xi}) - \mathbf{T}_i(\boldsymbol{\zeta})) \cdot (\boldsymbol{\xi} - \boldsymbol{\zeta}) > 0 \quad \forall \boldsymbol{\xi}, \boldsymbol{\zeta} \in \mathbb{R}^{d^2}, \boldsymbol{\xi} \neq \boldsymbol{\zeta}. \tag{1.5}$$

The motivation for considering the simplified cartoon given in (1.3) comes from the recent article [2], where Anand and Rajagopal discuss and model the influence of platelet activation on blood rheology. Despite the fact that platelets constitute only small portion of the blood, they are extremely sensitive to chemical and mechanical changes. At high shear rates (or high shear stresses), platelets release carried chemical species and a set off chemical reactions. This results in the formation of platelet aggregates that exhibit significantly different characteristics than the blood did before the platelet activation process started. In [2] Anand and Rajagopal propose a constitutive equation for blood, in the framework of rate-type (viscoelastic) incompressible fluid-like materials, which is capable of incorporating platelet activation resulting in distinctly different material moduli (i.e. the viscosity, relaxation times, etc.) before and after the activation.

The constitutive equation (1.3) simplifies the model proposed by Anand and Rajagopal in several respects. First of all, we completely neglect the elastic response exhibited by blood due to the presence of red blood cells, white blood

[2] The flow of glaciers, sand, or any other densely packed granular material (modeled as a single continuum) can serve as a good example.