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A predictor–corrector scheme based on the ADI method for pricing American puts with stochastic volatility

Song-Ping Zhu *, Wen-Ting Chen

School of Mathematics and Applied Statistics, University of Wollongong NSW 2522, Australia

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ABSTRACT

In this paper, we introduce a new numerical scheme, based on the ADI (alternating direction implicit) method, to price American put options with a stochastic volatility model. Upon applying a front-fixing transformation to transform the unknown free boundary into a known and fixed boundary in the transformed space, a predictor–corrector finite difference scheme is then developed to solve for the optimal exercise price and the option values simultaneously. Based on the local von Neumann stability analysis, a stability requirement is theoretically obtained first and then tested numerically. It is shown that the instability introduced by the predictor can be damped, to some extent, by the ADI method that is used in the corrector. The results of various numerical experiments show that this new approach is fast and accurate, and can be easily extended to other types of financial derivatives with an American-style exercise.

Another key contribution of this paper is the proposition of a set of appropriate boundary conditions, particularly in the volatility direction, upon realizing that appropriate boundary conditions in the volatility direction for stochastic volatility models appear to be controversial in the literature. A sound justification is also provided for the proposed boundary conditions mathematically as well as financially.

1. Introduction

It is well known that one of the most important topics in quantitative finance research is the valuation of option derivatives. Empirical evidence suggests that the Black–Scholes model, which is a breakthrough in the financial area, is inadequate to describe asset returns and the behavior of the option markets [1]. This is because their assumption on the log-normality of the value of the underlying asset has somewhat oversimplified the real process of the asset price. One possible remedy is to assume that the volatility of the asset price also follows a stochastic process [2–5]. In this paper, we will use the stochastic model introduced by Heston for pricing American options [4]. In this model, it assumes that the variance (the square of the underlying price volatility) follows a random process known in financial literature as the Cox–Ingersoll–Ross (CIR) process and in mathematical statistics as the Feller process [3,6]. Empirical studies suggest that this non-negative and mean-reverting process is indeed more consistent with what has been observed in real markets [7–9]. For example, Adrian and Victor [1] showed that the time-dependent probability distribution of the changes of the stock index generated in the Heston model agrees well with the Dow-Jones data after the calibration of the parameters in this model.

How to rationally price an option remains one of the major challenges in today's finance industry. This is even for so for pricing American options as the challenge stems from the nonlinearity originated from the inherent characteristics that an American option can be exercised at any time during its lifespan and thus the additional right of being able to exercise the option early, in comparison with a European option, casts the problem into a free boundary problem, which is far more difficult to deal with, even under the traditional Black–Scholes framework. In this area, there have been predominantly two kinds of approaches, numerical methods and analytical approximations, for the valuation of American

* Corresponding author. Fax: +61 2 42214845.
E-mail addresses: spz@uow.edu.au (S.-P. Zhu), wc904@uow.edu.au (W.-T. Chen).

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options under the Black–Scholes framework. While the former typically includes the finite difference method [10,11], the finite element method [12], the binomial method [13] and the Monte Carlo simulation method [14], the latter includes the Richardson extrapolation approximation [15], the Laplace transform method [16], the algebraic equation method [17,18] and the integral-equation method [19–22]. Furthermore, recently, an analytical solution for American puts on a non-dividend underlying asset was even found out by Zhu [23]. It must be pointed out that all these approaches cannot be easily extended to the Heston model, primarily due to the fact that, under stochastic volatility, the optimal exercise price depends, in addition to time, on the dynamics of volatility. In other words, the introduction of a second stochastic process has considerably complicated the solution process in pricing American options.

In the last decade, several numerical approaches based on the finite difference method (FDM) are introduced to solve the free boundary problem associated with the valuation of American options under the Heston model. For instance, Clarke and Parrott [24], used a special version of a projected full approximation scheme with multigrid to solve the American option pricing problem. One advantage of such a multilevel method is that the number of iterations required to solve a linear complementarity problem is essentially independent of the grid size. However, their method is rather complicated because of the use of a special projected linear Gauss–Seidel smoother. Ikonen and Toivanen [25] calculated the option values by using the operator splitting method, in which an auxiliary variable is used to improve the accuracy. However, their method still requires a relatively large amount of computational storage space. Zvan et al. [26] applied the penalty method to the American option pricing problem. Their method is simpler than the one used by Clark and Parrott, but still needs a relatively large amount of computational resources to produce an accurate result.

Since most of the numerical methods in the literature are either too complicated to implement or with very low computational efficiency, it is desirable to have alternative ways to deal with the valuation of American options with stochastic volatility. In this paper, we propose an approach based on a predictor–corrector framework, which is commonly used to numerically solve nonlinear partial differential equations (PDEs). The idea behind the predictor–corrector method is to use a suitable combination of an explicit and an implicit technique to obtain a method with better convergence characteristics. Previously, this scheme has only been applied to the pricing problem under the Black–Scholes model, such as Zhu and Zhang reported in [11]. Though their method is efficient and accurate, it cannot be applied directly to the stochastic models. The purpose of this article is to introduce a new predictor–corrector scheme, which is not only suitable for the Heston model, but also for other stochastic models. In our new approach, we adopt the so-called front-fixing transformation [27] to let the unknown boundary be included in the governing equation as a nonlinear term in exchange for a fixed boundary. To tackle the nonlinear nature of American option pricing problem, which is explicitly exposed in the transformed equation, we use a predictor–corrector finite difference scheme at each time step to convert the nonlinear PDE to two linearized difference equations associated with the prediction and correction phase, respectively. The prediction phase, constructed by an explicit Euler scheme, is used to calculate the optimal exercise price, whereas the correction phase, designed by the alternating direction implicit, or ADI, method, continues to do the calculation of the option price together with the correction of the optimal exercise price. The ADI scheme used in the corrector is efficient in computing multi-dimensional problems. Moreover, it is also suggested that the good convergence property of the ADI scheme can somehow, damp the instability that might be introduced by the predictor. With the perfect combination of the explicit Euler scheme and the ADI method, the originally nonlinear problem has been successfully converted to a set of linear algebraic equations, which can be solved efficiently. In comparison with the numerical methods in the literature, the advantage of the current scheme is obvious. For example, first, our method requires almost the same storage space as a one-dimensional problem does and it will not increase even when the method is applied to option pricing problems on multi-assets. This is, however, not the case for the numerical methods proposed in [25], as a substantially larger amount of the storage space is required, which will also increase as the number of the assets increases. Second, in addition to the option values, the present method captures the entire optimal exercise boundary as part of the solution procedure, whereas in [25], the optimal exercise price cannot be obtained simultaneously, and needs to be solved with some extra effort. Finally, our method requires no iterations, and can be easily extended to the valuation of American options under other models.

It is usually easy to design a numerical scheme to solve a PDE system, but much harder to provide a theoretical threshold for the stability and the convergence of the scheme. It is probably even worse to theoretically define a suitable stability criterion for the predictor–corrector method, since it is a hybrid finite difference method. For this reason, the issue of stability requirement was not even attempted in [11] for the Black–Scholes case. One could naturally imagine that with the complexity of the Heston model, it would have made a theoretical stability analysis much less achievable. Based on the local von Neumann stability analysis, combining with the ‘‘frozen’’ coefficient technique, which is commonly used for the stability analysis of the variable-coefficient problem [28], we have managed to not only verify that the ADI method used for the European puts under the Heston model is unconditionally stable, but also give a proper stability requirement for the predictor–corrector approach.

In the subsequent sections, we will present this new approach together with the numerical results for American put options under the Heston model. The paper is organized as follows: in Section 2, we introduce the PDE system that the price of an American put must satisfy under the Heston model, with our emphasis being placed on properly closing the system with appropriate boundary conditions, which appear to be controversial in the literature. In Section 3, we present our predictor–corrector approach in detail as well as the implementation of the ADI scheme. In Section 4, numerical examples and some analyses are presented to demonstrate the convergence and accuracy of the current scheme. Concluding remarks are given in the last section.