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Characteristic functions and option valuation in a Markov chain market

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ABSTRACT

We introduce an approach for valuing some path-dependent options in a discrete-time Markov chain market based on the characteristic function of a vector of occupation times of the chain. A pricing kernel is introduced and analytical formulas for the prices of Asian options and occupation time call options are derived.

1. Introduction

The valuation of options has long been an important issue in financial economics. The history of this problem may be traced to the early work of Bachelier [1], where movements of share prices were modeled by an arithmetic Brownian motion and an option valuation formula derived. This important piece of work was re-discovered by Paul Samuelson in the 1960s and re-generated interest in option valuation. The pioneering works of Black and Scholes [2] and Merton [3] provided a solution to option valuation and hedging. Under the geometric Brownian motion assumption for the price process of the underlying share price, the assumption of a perfect market and the no-arbitrage assumption, they derived a closed-form pricing formula for the price of a standard European call option. Since the works of Black and Scholes [2] and Merton [3], there has been tremendous growth in both academic and practical research on option valuation and hedging, as well as the related trading activities of derivative securities in global financial markets. Coincidentally, the Chicago Board of Trade (CBOT) started trading standardized call option contracts in 1973.

A key insight of the Black–Scholes–Merton option pricing theory is the use of risk-neutral valuation, where the appreciation rate of the underlying share is replaced by the risk-free rate of interest and the pricing is then accomplished in the risk-neutral world. This procedure becomes transparent in the discrete-time binomial option valuation model introduced by Cox et al. [4].[1] Besides giving a transparent relationship between risk-neutral valuation and no arbitrage, the binomial, or CRR, option valuation model also provides a simple and efficient numerical scheme to approximate option prices in a continuous-time model. Assuming that the share price takes one of two possible values in each period may not be accurate enough to describe real-world movements of share prices, so more complicated tree structures for option valuation have been proposed in the literature. Boyle [5] proposed a trinomial lattice model. As an extension of the binomial model, the trinomial lattice model assumes that the price of the underlying share over each time period may take one of

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[1] Indeed, this idea of binomial asset price model was originated by William Sharpe.

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three possible values. Indeed, the trinomial lattice model was motivated by a finite-difference numerical scheme for solving partial differential equations. He [6] proposed a multi-nominal option valuation model, while preserving completeness of the market.

Discrete-time Markov chain models provide an important class of asset price models. They have been considered by authors such as Pliska [7], Norberg [8] and van der Hoek and Elliott [9]. Some related models include Song et al. [10] for a multivariate Markov chain asset price models and Valakevicius [11] for a continuous-time Markov chain asset price models. One of the key motives for considering Markov chain asset price models is that the discrete-time Markov chain can provide a reasonable approximation to continuous-time diffusion processes. Indeed, Markov chain asset price models may include binomial and trinomial asset price models as particular cases. The valuation of some exotic options may be more simple in a discrete-time Markov chain asset price model.

In this paper, we introduce a characteristic function approach for the valuation of some path-dependent options, such as Asian options and occupation time options, in a Markov chain market, where uncertainty is modeled by a discrete-time, finite-state, Markov chain. A characteristic function of a vector of occupation times of the chain over different states is defined, which is the key tool for valuing the options. We also discuss the issue of selecting a pricing kernel in such a Markov chain market. Analytical formulas for the prices of Asian options and occupation time call options are then derived.

This paper is organized as follows. Section 2 presents the Markov chain market and the price dynamics. In Section 3, we derive the characteristic function of a vector of occupation times in different states of the underlying Markov chain. Section 4 discusses the choice of a pricing kernel in the Markov chain market. We derive analytical pricing formulas for an arithmetic Asian option, a geometric Asian option and an occupation time call option using the characteristic function derived in Section 3. The final section gives a summary of the paper.

2. A Markov chain market model

In this section, we present a discrete-time Markov chain market model, where the randomness of the price process of a share is modeled by a discrete-time, finite-state, time-homogeneous, Markov chain. The Markov chain asset price model considered here includes the trinomial asset price model as a special case as explained later in this section. Indeed, similar models were discussed in some recent work such as Valakevicius [11], Song et al. [10] and van der Hoek and Elliott [9].

We consider a complete probability space $(\Omega, \mathcal{F}, P)$, where P is a real-world probability measure. Let $\mathcal{T} := \{0, 1, 2, \dots, T\}$ be the time parameter set, where T is a finite positive integer. Indeed, one may consider an infinite time parameter set. However, for our purpose, a finite time parameter set is enough. We suppose that the risk-free interest rate is a constant $r \in (0, 1)$.

To describe uncertainty or randomness in the Markov chain market, we consider a discrete-time, N-state, time-homogeneous Markov chain $\{\mathbf{X}_t\}_{t\in\mathcal{T}}$. Following the convention in [12], we identify the state space of the chain $\{\mathbf{X}_t\}_{t\in\mathcal{T}}$ with the canonical state space given by the set of standard unit vectors in $\mathbb{R}^N$:

$$\mathcal{E} = \{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_N\}.$$

Here, for each $i = 1, 2, \dots, N$, $\mathbf{e}_i$ is the unit vector in $\mathbb{R}^N$ with one as the i-th element and zeros elsewhere. That is, $\mathbf{e}_i := (0, \dots, 1, \dots, 0)'$ with $\mathbf{x}'$ the transpose of a vector $\mathbf{x}$.

To describe the probability law of the chain, we define the following transition probabilities and transition matrix:

$$a_{ji} = P(\mathbf{X}_{t+1} = \mathbf{e}_j \mid \mathbf{X}_t = \mathbf{e}_i),$$

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1N} \\ a_{21} & a_{22} & \cdots & a_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ a_{N1} & a_{N2} & \cdots & a_{NN} \end{pmatrix}.$$

Define the martingale increment process $\{\mathbf{V}_t\}$ by

$$\mathbf{V}_{t+1} := \mathbf{X}_{t+1} - \mathbf{A}\mathbf{X}_t$$

so

$$\mathrm{E}[\mathbf{V}_{t+1} \mid \mathcal{F}_t] = \mathbf{0} \in \mathbb{R}^N.$$

Here $\{\mathcal{F}_t\}_{t\in\mathcal{T}}$ is the natural filtration generated by the Markov chain.

We now define a share price process $\{S_t\}_{t\in\mathcal{T}}$ by assuming that it can only take values from a finite set of values $\mathcal{S} = \{s_1, s_2, \dots, s_N\} \subset [0, \infty)$. Write

$$\mathbf{s} := (s_1, s_2, \dots, s_N)'.$$

Without loss of generality, we suppose that $0 \le s_1 < s_2 < \cdots < s_N$. Then in our model, the share price process $\{S_t\}$ is governed by the Markov chain $\{\mathbf{X}_t\}$ by means of the definition:

$$S_t = \langle \mathbf{s}, \mathbf{X}_t \rangle.$$

Consequently, the price process $\{S_t\}$ is, again, a discrete-time, finite-state Markov chain. Here, $\langle \cdot, \cdot \rangle$ is the scalar product.