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New existence of homoclinic orbits for a second-order Hamiltonian system ☆

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ABSTRACT

By using the Mountain Pass Theorem and the Symmetric Mountain Pass Theorem, we establish some existence criteria to guarantee that the second-order Hamiltonian system $\ddot{u}(t) - a(t)|u(t)|^{p-2}u(t) + \nabla W(t, u(t)) = 0$ has at least one or infinitely many homoclinic orbits, where $t \in \mathbb{R},\ u \in \mathbb{R}^N$, $a \in C(\mathbb{R}, \mathbb{R})$ and $W \in C^1(\mathbb{R} \times \mathbb{R}^N, \mathbb{R})$ are not periodic in t. Our conditions on the potential $W(t, x)$ are rather relaxed.

1. Introduction

The goal of this paper is to prove the existence and multiplicity for infinitely many homoclinic orbits from 0 of the second-order Hamiltonian system

$$\ddot{u}(t) - a(t)|u(t)|^{p-2}u(t) + \nabla W(t, u(t)) = 0, \tag{1.1}$$

where $p \geq 2,\ t \in \mathbb{R},\ u \in \mathbb{R}^N$, $a : \mathbb{R} \to \mathbb{R}$ and $W : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$. As usual, we say that a solution $u(t)$ of (1.1) is homoclinic (to 0) if $u(t) \to 0$ as $t \to \pm\infty$. In addition, if $u(t) \not\equiv 0$ then $u(t)$ is called a nontrivial homoclinic solution.

It is well known that homoclinic orbits play an important role in analyzing the chaos of dynamical systems. If a system has the transversely intersected homoclinic orbits, then it must be chaotic. If it has the smoothly connected homoclinic orbits, then it cannot stand the perturbation, its perturbed system probably produce chaotic phenomena. Therefore, it is of practical importance and mathematical significance to consider the existence of homoclinic orbits of (1.1) emanating from 0.

When $p = 2$, system (1.1) reduces to the following second-order Hamiltonian system

$$\ddot{u}(t) - a(t)u(t) + \nabla W(t, u(t)) = 0. \tag{1.2}$$

In recent years, the existence and multiplicity of homoclinic orbits for Hamiltonian systems have been investigated in many papers via variational methods and many results were obtained based on various hypotheses on the potential functions; see, e.g., [1–19]. For system (1.2), if $a(t)$ and $W(t, x)$ are T-periodic in t, Rabinowitz [14] showed the existence of homoclinic orbits as a limit of $2kT$-periodic solutions of system (1.2).

If $a(t)$ and $W(t, x)$ are not periodic in t, the problem of existence of homoclinic orbits for system (1.2) is quite different from the ones just described, because of lack of compactness of the Sobolev embedding. In [15], Rabinowitz and Tanaka

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studied (1.2) without a periodicity assumption, by using a variant of the Mountain Pass Theorem without the Palais–Smale condition.

However, to our best knowledge, few results are obtained in the literature for (1.1). The traditional ways to establish the variational structure in [20,15,21,18] are inapplicable to our case. Salvatore dealt with the homoclinic orbits of the system (1.1) when $p > 2$ by different methods and obtained the following results by introducing suitable weighted Sobolev space; see [22].

Theorem A ([22]). *Assume that a and W satisfy the following conditions:* (A) *Let $p > 2$, $a(t)$ is a continuous, positive function on $\mathbb{R}$ such that for all $t \in \mathbb{R}$*

$$a(t) \geq \gamma |t|^{\alpha}, \quad \alpha > \frac{p-2}{2}, \ \gamma > 0;$$

(W1) *$W \in C^1(\mathbb{R} \times \mathbb{R}^N, \mathbb{R})$ and there is a constant $\mu > p$ such that*

$$0 < \mu W(t,x) \leq (\nabla W(t,x), x), \quad \forall (t,x) \in \mathbb{R} \times \mathbb{R}^N \setminus \{0\}.$$

(W2) *$|\nabla W(t,x)| = o(|x|^{p-1})$ as $|x| \to 0$ uniformly with respect to $t \in \mathbb{R}$.*
(W3) *There is a $\overline{W} \in C(\mathbb{R}^N, \mathbb{R})$ such that*

$$|W(t,x)| + |\nabla W(t,x)| \leq |\overline{W}(x)|, \quad \forall (t,x) \in \mathbb{R} \times \mathbb{R}^N.$$

Then system (1.1) possesses a nontrivial homoclinic solution.

When $W(t,x)$ is an even function on x, Salvatore [22] still used the Symmetric Mountain Pass Theorem to prove the following theorem on the existence of an unbounded sequence of homoclinic orbits of system (1.1).

Theorem B ([22]). *Assume that a and W satisfy* (A), (W1)–(W3) *and the following condition:*

(W4) $W(t,-x) = W(t,x), \ \forall (t,x) \in \mathbb{R} \times \mathbb{R}^N$.

Then there exists an unbounded sequence of homoclinic solutions for system (1.1).

Motivated by Salvatore [22,21], Tang [16], the aim of this paper is to find homoclinic solutions under some relaxed assumptions on $W(t,x)$. Indeed, we establish some existence criteria to guarantee that system (1.1) has at least one or infinitely many homoclinic solutions by using the Mountain Pass Theorem or the Symmetric Mountain Pass Theorem. In particular, our results generalize Theorems A and B by relaxing condition (W1) and (W2) and removing condition (W3), which has not been often considered in the literature.

Our main results are the following theorems.

Theorem 1.1. *Assume that a and W satisfy* (A) *and the following assumptions:*

(W5) *$W(t,x) = W_1(t,x) - W_2(t,x)$, $W_1, W_2 \in C^1(\mathbb{R} \times \mathbb{R}^N, \mathbb{R})$, and there is $R > 0$ such that*

$$\frac{1}{a(t)} |\nabla W(t,x)| = o(|x|^{p-1}) \quad \text{as } x \to 0$$

uniformly in $t \in (-\infty, -R] \cup [R, +\infty)$.
(W6) *There is a constant $\mu > p$ such that*

$$0 < \mu W_1(t,x) \leq (\nabla W_1(t,x), x), \quad \forall (t,x) \in \mathbb{R} \times \mathbb{R}^N \setminus \{0\}.$$

(W7) *$W_2(t,0) \equiv 0$ and there is a constant $\varrho \in (p, \mu)$ such that*

$$W_2(t,x) \geq 0, \qquad (\nabla W_2(t,x), x) \leq \varrho W_2(t,x), \quad \forall (t,x) \in \mathbb{R} \times \mathbb{R}^N.$$

Then system (1.1) possesses a nontrivial homoclinic solution.

Theorem 1.2. *Assume that a and W satisfy* (A), (W6) *and the following assumptions:*

(W5′) *$W(t,x) = W_1(t,x) - W_2(t,x)$, $W_1, W_2 \in C^1(\mathbb{R} \times \mathbb{R}^N, \mathbb{R})$, and*

$$\frac{1}{a(t)} |\nabla W(t,x)| = o(|x|^{p-1}) \quad \text{as } x \to 0$$

uniformly in $t \in \mathbb{R}$.
(W7′) *$W_2(t,0) \equiv 0$ and there is a constant $\varrho \in (p, \mu)$ such that*

$$(\nabla W_2(t,x), x) \leq \varrho W_2(t,x), \qquad \forall \, (t,x) \in \mathbb{R} \times \mathbb{R}^N.$$

Then system (1.1) possesses a nontrivial homoclinic solution.