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Neighborhoods of a new class of harmonic multivalent functions

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ABSTRACT

We introduce and investigate a new subclass of harmonic multivalent functions defined by using a differential operator. We obtain coefficient conditions, distortion bounds, extreme points, convex combination for the above class of harmonic multivalent functions. We also, derive inclusion relationships involving the neighborhoods of harmonic multivalent functions.

1. Introduction

A continuous function $f = u + iv$ is a complex valued harmonic function in a domain $D \subset \mathbb{C}$ if both u and v are real harmonic in D. In any simply connected domain D we can write $f = h + \overline{g}$, where h and g are analytic in D. We call h the analytic part and g the co-analytic part of f. The harmonic function $f = h + \overline{g}$ is sense preserving and locally one to one in D if $|h'(z)| > |g'(z)|$ in D. See Clunie and Sheil-Small [1].

For $p \geq 1$, $n \in \mathbb{N}$, denote by $SH\ (n, p)$ the class of functions $f = h + \overline{g}$ that are sense preserving, harmonic multivalent in the unit disk $U = \{z : |z| < 1\}$, where h and g defined by

$$h(z) = z^p + \sum_{k=n+p}^{\infty} a_k z^k, \qquad g(z) = \sum_{k=n+p-1}^{\infty} b_k z^k,\ |b_{n+p-1}| < 1, \tag{1.1}$$

which are analytic and multivalent functions in U.

Note that $SH(n, p)$ reduces to $S(n, p)$, the class of analytic multivalent functions, if the co-analytic part of $f = h + \overline{g}$ is identically zero.

Let $f^{(q)}$ denote the qth-order ordinary differential operator, for a function $f \in SH\ (n, p)$, that is,

$$f^{(q)}(z) = h^{(q)}(z) + \overline{g^{(q)}(z)}$$

where $h^{(q)}(z) = \frac{p!}{(p-q)!} z^{p-q} + \sum_{k=n+p}^{\infty} \frac{k!}{(k-q)!} a_k\, z^{k-q}$ and $g^{(q)}(z) = \sum_{k=n+p-1}^{\infty} \frac{k!}{(k-q)!} b_k\, z^{k-q}$, $p > q, p \in \mathbb{N}, q \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, n \in \mathbb{N}, z \in U$.

Next, $D^m f^{(q)}(z)$ is defined by

$$D^m f^{(q)}(z) = D^m h^{(q)}(z) + (-1)^m \overline{D^m g^{(q)}(z)} \tag{1.2}$$

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where

$$D^m h^{(q)}(z) = \frac{(p-q)^m}{(p-q)!} p! z^{p-q} + \sum_{k=n+p}^{\infty} \frac{(k-q)^m}{(k-q)!} k! a_k z^{k-q} \quad \text{and}$$

$$D^m g^{(q)}(z) = \sum_{k=n+p-1}^{\infty} \frac{(k-q)^m}{(k-q)!} k! b_k z^{k-q}, \quad m \in \mathbb{N}_0, z \in U.$$

In view of (1.2), it is clear that

$$D^0 f^{(0)}(z) = h(z) + \overline{g(z)}$$
$$D^0 f^{(1)}(z) = h'(z) + \overline{g'(z)}$$
$$D^1 f^{(0)}(z) = zh'(z) - \overline{zg'(z)}.$$

For $q = 0$, the differential operator $D^m f^{(q)}$ was introduced for the class $S(1,1)$ by Salagean [2] and modified for the class $SH(1,1)$ by Jahangiri et al. [3].

We will use the notations

$$\frac{(i-q)^j}{(i-q)!} i! := q_i^j, \quad i = p, k, n+p, n+p-1 \text{ and } j = m, m+1.$$

Let

$$F(z) = (1-\lambda) D^m f^{(q)}(z) + \lambda D^{m+1} f^{(q)}(z) = H(z) + \overline{G(z)} (f(z) \in SH(n,p), 0 \leq \lambda \leq 1)$$

where H and G are of the form

$$H(z) = (1 - \lambda + \lambda(p-q)) q_p^m z^{p-q} + \sum_{k=n+p}^{\infty} (1 - \lambda + \lambda(k-q)) q_k^m a_k z^{k-q},$$

$$G(z) = (-1)^m \sum_{k=n+p-1}^{\infty} ((1-\lambda) - \lambda(k-q)) q_k^m b_k z^{k-q}. \tag{1.3}$$

Let $SH_{n,p}^m(q, \lambda, \alpha)$ denote the subclass of $SH(n,p)$ consisting of functions $f = h + \overline{g} \in SH(n,p)$ that satisfy the condition

$$\operatorname{Re}\left\{\frac{zH'(z) - \overline{zG'(z)}}{H(z) + \overline{G(z)}}\right\} > \alpha(p-q), \quad (0 \leq \alpha < 1, p > q, p \in \mathbb{N}, q \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, n \in \mathbb{N}, z \in U) \tag{1.4}$$

where $H(z)$ and $G(z)$ are given by (1.3).

Denote by $\overline{SH}(n,p)$ the subclass of $SH(n,p)$, consisting of harmonic functions $f_m = h + \overline{g_m}$ where h and g_m are of the form

$$h(z) = z^p - \sum_{k=n+p}^{\infty} a_k z^k, \; g_m(z) = (-1)^m \sum_{k=n+p-1}^{\infty} b_k z^k, \; a_k, b_k \geq 0. \tag{1.5}$$

Define $\overline{SH}_{n,p}^m(q, \lambda, \alpha) := SH_{n,p}^m(q, \lambda, \alpha) \cap \overline{SH}(n,p)$.

The classes $SH_{n,p}^m(q, \lambda, \alpha)$ and $\overline{SH}_{n,p}^m(q, \lambda, \alpha)$ include a variety of well-known subclasses of $SH(n,p)$. For example,

(i) $SH_{1,1}^0(0,0,0) \equiv SH^*$ is the class of sense-preserving, harmonic univalent functions f which are starlike in U (see [4,5]);

(ii) $\overline{SH}_{1,1}^0(0,0,\alpha) \equiv SH^*(\alpha)$ is the class of sense-preserving, harmonic univalent functions f which are starlike of order α in U (see [6]);

(iii) $\overline{SH}_{1,1}^1(0,0,\alpha) \equiv HK(\alpha)$ is the class of sense-preserving, harmonic univalent functions f which are convex of order α in U (see [4]);

(iv) $\overline{SH}_{1,p}^0(0,0,0) \equiv SH^*(p)$ is the class of sense-preserving, harmonic multivalent functions which are starlike in U (see [7]);

(v) $\overline{SH}_{1,1}^m(0,0,\alpha) \equiv \overline{H}(m,\alpha)$ is the class of sense preserving, Salagean-type harmonic univalent functions in U (see [3]);

(vi) $\overline{SH}_{1,1}^m(0,0,\alpha) \equiv \overline{S}_H(m+1, m; \alpha)$ (see [8]).

If the co-analytic part of $f = h + \overline{g}$ of the form (1.1) is identically zero and specialize the parameters, we obtain the following subclasses:

(i) $\overline{SH}_{n,p}^m(q,0,\alpha) \equiv S_{n,p}^m(q, (1-\alpha)(p-q), 1)$ (see [9]);

(ii) $\overline{SH}_{n,p}^0(q,0,\alpha) \equiv S_{n,p}^q(0, 1, (1-\alpha)(p-q))$ (see [10]).