Computers and Mathematics with Applications 65 (2013) 673–681

Contents lists available at SciVerse ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

Constructing PDE-based surfaces bounded by geodesics or lines of curvature

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ARTICLE INFO

Article history:
Received 1 July 2012
Received in revised form 14 October 2012
Accepted 20 November 2012

Keywords:
Partial differential equations
Geodesic
Line of curvature
Surface modeling
Biharmonic surfaces

ABSTRACT

In order to explore a new approach to construct surfaces bounded by geodesics or lines of curvature, a method of surface modeling based on fourth-order partial differential equations (PDEs) is presented. Compared with the free-form surface modeling based on finding control points, PDE-based surface modeling has the following three advantages. First, the corresponding biharmonic surface can naturally be derived under some degenerative conditions; second, the parameters in the PDE implicate some physical meaning, such as elasticity or rigidity; third, there are only a few parameters that need to be evaluated, and hence the computation is simple. In addition, this paper constructs two adjacent surfaces with C^1 continuity whose common boundary is the same given curve as well as respective geodesic (or line of curvature). Examples show that this method to construct PDE-based surfaces bounded by geodesics or lines of curvature is easy and effective.

1. Introduction

Surface modeling is a central issue in computer-aided design, and surfaces are usually represented by implicit or parametric forms. For example, [1] transformed the construction of a surface into Poisson problems and then obtained implicit surfaces. Compared with implicit surfaces, parametric surfaces are used more widely, and the most usual modeling technique is to construct free-form parametric surfaces based on determining control points. [2] constructed parametric spline surfaces by using variable-degree polynomial splines. [3] constructed harmonic and homogenous biharmonic Bézier surfaces bounded by the given curves. [4] constructed nonhomogenous biharmonic and tetraharmonic Bézier surfaces according to given boundary curves and tangent conditions along them. In addition, for free-form parametric surfaces, there is another modeling technique that needs to be paid attention to, which is the parametric surface construction based on PDEs. [5] constructed parametric surfaces using a fourth-order PDE with three shape control parameters, and presented exact analytic solutions of the PDE in some cases. [6] constructed surfaces using a sixth-order PDE, which interpolate the given boundary curves, boundary tangents, and boundary curvatures. In recent years, people have become interested in two additional problems in surface modeling. One is to construct surfaces bounded by given geodesic curves, and the other is to construct surfaces bounded by lines of curvature.

A geodesic curve is intrinsic to the geometric characterization of surfaces. Geodesics are used in many fields. For example, they are used in object segmentation [7,8] and multi-scale image analysis [9], in computer vision and image processing. In surface modeling, [10] constructed parametric surfaces bounded by the given geodesic curves; [11] constructed polynomial ruled surfaces with the given boundary as geodesic curves; similarly, [12] constructed cubic polynomial ruled patches with the given geodesic boundary. Recently, [13] constructed Bézier surfaces bounded by the four given geodesic curves, and [14]

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doi:10.1016/j.camwa.2012.11.018

constructed triangular Bézier surfaces using the methods in [13]. As we know, the existed method of constructing parametric surfaces bounded by geodesic curves is usually based on finding corresponding control points, but a method based on PDEs has not yet been published in the literature.

On the other hand, lines of curvature have important applications in computer graphics and product manufacturing. [15] studied an approach to compute lines of curvature near the umbilical points on the surfaces, and applied it to extracting the generic features of free-form parametric surfaces for shape interrogation. [16] presented a method to determine lines of curvature in point cloud models, and applied it to reconstruct meshes that interpolate the given lines of curvature. [17] showed an application of lines of curvature in plate-metal-based manufacturing. In the surface modeling literature, there are only a few articles concerning lines of curvature. Recently, [18] constructed Bézier parametric surfaces bounded by the four lines of curvature, but the construction of a PDE-based surface bounded by lines of curvature has not been studied in the literature.

Hence, this paper aims to address the issue of constructing a PDE-based surface bounded by geodesics or lines of curvature. The work of this paper can be listed as follows. First, the method to generate a surface bounded by geodesics or lines of curvature based on a vector-valued fourth-order PDE is presented, and the exact solutions of the PDE for some given conditions are obtained by using the method in [5]. Second, the numerical solution of the PDE by the least-square method is given. Third, the process shows that a fourth-order PDE-based surface can be degenerated into a biharmonic surface by choosing appropriate coefficients. Fourth, two adjacent surfaces are constructed so that they have a common boundary which is the same given curve as well as the respective geodesic, and the continuity between them is also discussed.

The rest of the paper is arranged as follows. Section 2 presents the construction of surfaces bounded by geodesic curves. Section 3 presents the construction of surfaces bounded by lines of curvature, and the conclusion is presented in the last section.

2. Construction of surfaces bounded by geodesics

2.1. Description of surfaces with geodesic boundary curves

Suppose that the curves $\boldsymbol{p}_0(u), \boldsymbol{p}_1(u), u \in [a, b]$ are given, where a, b are both arbitrarily real numbers. Then the surface

$$\boldsymbol{p}(u, v) = (x(u, v), y(u, v), z(u, v))^T$$

with the given curves as boundary curves should satisfy

$$\boldsymbol{p}(u, 0) = \boldsymbol{p}_0(u), \quad (1)$$

$$\boldsymbol{p}(u, 1) = \boldsymbol{p}_1(u). \quad (2)$$

In order to make the given curves $\boldsymbol{p}_0(u), \boldsymbol{p}_1(u)$ be geodesics on the surface $\boldsymbol{p}(u, v)$, the necessary condition is that the vectors $\boldsymbol{b}_0(u) = \boldsymbol{p}_0'(u) \times \boldsymbol{p}_0''(u)$ and $\boldsymbol{b}_1(u) = \boldsymbol{p}_1'(u) \times \boldsymbol{p}_1''(u)$ are all tangent to the surface. The vectors $\boldsymbol{b}_0(u)$ and $\boldsymbol{b}_1(u)$ are parallel to the binormals of the curves $\boldsymbol{p}_0(u)$ and $\boldsymbol{p}_1(u)$, respectively. Specifically, the partial derivative with respect to v of the surface should satisfy the following equations:

$$\boldsymbol{p}_v(u, 0) = \alpha_0(u)\boldsymbol{p}_0'(u) + \beta_0(u)\boldsymbol{b}_0(u), \quad (3)$$

$$\boldsymbol{p}_v(u, 1) = \alpha_1(u)\boldsymbol{p}_1'(u) + \beta_1(u)\boldsymbol{b}_1(u), \quad (4)$$

where $\alpha_i(u), \beta_i(u) \neq 0, i = 0, 1$ are any functions defined on $[a, b]$. In addition, suppose that the surface satisfies the following fourth-order PDE:

$$\left(\boldsymbol{a}\frac{\partial^4}{\partial u^4} + \boldsymbol{b}\frac{\partial^4}{\partial u^2 \partial v^2} + \boldsymbol{c}\frac{\partial^4}{\partial v^4}\right)\boldsymbol{p}(u, v) = 0, \quad (5)$$

where $\boldsymbol{a} = (a_x, a_y, a_z)^T, \boldsymbol{b} = (b_x, b_y, b_z)^T, \boldsymbol{c} = (c_x, c_y, c_z)^T$ are vectors with positive components.

The PDE in Eq. (5) includes all forms of the existing fourth-order PDEs used for surface generation [5], and another advantage of this PDE is that the surface $\boldsymbol{p}(u, v)$ becomes a biharmonic surface if we choose the coefficients as $\boldsymbol{a} = (1, 1, 1)^T, \boldsymbol{b} = (2, 2, 2)^T, \boldsymbol{c} = (1, 1, 1)^T$. So the surface satisfying Eq. (5) is interesting. A biharmonic surface is a surface that satisfies the equation $\Delta^2\boldsymbol{p}(u, v) = 0$, where $\Delta = \left(\frac{\partial}{\partial u^2} + \frac{\partial}{\partial v^2}\right)$. Besides, the parameters in the PDE imply some physical meaning. For example, a fourth-order PDE can be derived from the theory of bending thin elastic plates; thus the coefficients of such a PDE are closely related to the physical properties of the surface that it represents [19]. The parametric surface determined by Eqs. (1)–(5) is the one we need, i.e., the surface with the common boundary as respective geodesic.

Remark 1. The operations of the vectors defined in this paper are $\boldsymbol{ab} = (a_x b_x, a_y b_y, a_y b_y)^T$ and $\boldsymbol{a}/\boldsymbol{b} = (a_x/b_x, a_y/b_y, a_y/b_y)^T$, called vector multiplication or division, respectively, where $\boldsymbol{a} = (a_x, a_y, a_z)^T, \boldsymbol{b} = (b_x, b_y, b_z)^T$. In addition, $\boldsymbol{a} < \boldsymbol{b}$ means $a_x < b_x, a_y < b_y, a_z < b_z, 1 - \boldsymbol{a} = (1 - a_x, 1 - a_y, 1 - a_z), \xi\boldsymbol{a} = (\xi a_x, \xi a_y, \xi a_z), \sqrt{\frac{\boldsymbol{b}}{\boldsymbol{a}}} = \left(\sqrt{\frac{b_x}{a_x}}, \sqrt{\frac{b_y}{a_y}}, \sqrt{\frac{b_z}{a_z}}\right)$.